Machine Learning with Kernels for Portfolio Valuation and Risk Management

Damir Filipović
joint with Lotfi Boudabsa

École Polytechnique Fédérale de Lausanne
Swiss Finance Institute

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What this talk is about

- Develop a computational framework for quantitative portfolio risk management, based on dynamic value process
- Approximate and learn value process from a finite sample (learning from examples = machine learning) using kernel methods
- Rigorous implementation of kernel methods for simulation based approximation of functions

Main results

- Representer theorem yields closed form conditional expectations
- Central limit theorem in function space
- Finite sample guarantees thanks to weighted sampling
- Good quantitative results with relatively small training sample size
- "Regress-later" outperforms "regress-now"

Outline

Outline

Most economic scenario generators are of this form

- Economy with finite time horizon T
- $\mathbb{R}^d$-valued random driver $X = (X_1, \dots, X_T)$ with independent X_t's
- All financial values and cash flows discounted by some numeraire
- $\mathbb{Q}$ corresponding risk-neutral measure
- Physical measure $\mathbb{P} \sim \mathbb{Q}$

Dynamic value process

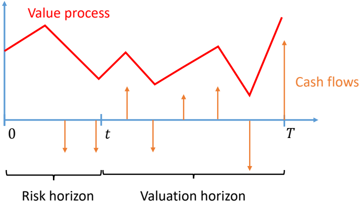
- **Objective:** a portfolio whose present value is not directly observable but has to be derived from its **cumulative cash flow** at T

$$f(X) = \sum_{t=1}^{T} \zeta_t$$

with time-t cash flows $\zeta_t = \zeta_t(X_1, \ldots, X_t) \in L^2_{\mathbb{Q}}$

- **Examples:** options, insurance liabilities, mortgage-backed instruments

- **Goal:** find **value process** of the portfolio, i.e., the $L^2_{\mathbb{Q}}$-martingale



$$V_t = \mathbb{E}_t^{\mathbb{Q}}[f(X)] := \mathbb{E}^{\mathbb{Q}}[f(X) \mid X_1, \ldots, X_t] = \underbrace{\sum_{s=1}^{t} \zeta_s}_{\text{cumulative CF at } t} + \underbrace{\mathbb{E}_t^{\mathbb{Q}}[\sum_{s=t+1}^{T} \zeta_s]}_{\text{time-}t\text{ value}}$$

Application: portfolio risk management

- **Risk measurement:** calculate $\mathbb{P}$-risk measure of 1-year P&L

$$\Delta V_{t+1} = V_{t+1} - V_t$$

E.g., (conditional) VaR or ES

- **Hedging:** given tradable financial instruments with gains processes G, i.e., $L^2_{\mathbb{Q}}$-martingales, find predictable hedging strategy ψ such that

$$\psi_t^\top \Delta G_t \approx \Delta V_t$$

E.g., variance minimizing

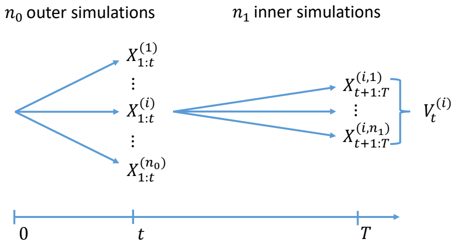
$$\psi_t = \mathbb{E}^{\mathbb{Q}}_{t-1}[\Delta G_t \Delta G_t^\top]^{-1} \mathbb{E}^{\mathbb{Q}}_{t-1}[\Delta G_t \, \Delta V_t]$$

- This is a **universal framework** for financial models (incl. GARCH)

Problem

How to compute value process $V_t = \mathbb{E}_t^{\mathbb{Q}}[f(X)]$, subject to

- lack of analytic solution: requires simulations



- limited computing budget: in realistic contexts train on small datasets, i.e., 1,000 to 32,000 paths
- Training with small datasets is vulnerable to overfitting

Machine learning approach—"Regress later"

1. **Approximate** f by some f_λ in $L^2_{\mathbb{Q}}$, for regularization parameter $\lambda > 0$
2. **Learn** f_λ from a finite sample $\boldsymbol{x} = (x(1), \dots, x(n))$, drawn from some sampling measure $\widetilde{\mathbb{Q}} \sim \mathbb{Q}$, along with function values $f(x(1)), \dots, f(x(n))$, gives sample estimator of the form

$$f_{\boldsymbol{x}}(x) = \sum_{j=1}^n \phi_j(x)\beta_j$$

such that $\mathbb{E}_t^{\mathbb{Q}}[\phi_j(X)]$ are given **in closed form**

Result: estimated value process **in closed form**:

$$V_{\boldsymbol{x},t} = \mathbb{E}_t^{\mathbb{Q}}[f_{\boldsymbol{x}}(X)], \quad t = 0, \dots, T$$

Basic estimation error bound

How good is this estimator? According to Doob's inequality, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[\max_{t\le T}|V_t - V_{\boldsymbol{x},t}|\right] &\le \|\tfrac{d\mathbb{P}}{d\mathbb{Q}}\|_{2,\mathbb{Q}}\,\mathbb{E}^{\mathbb{Q}}\left[\max_{t\le T}|V_t - V_{\boldsymbol{x},t}|^2\right]^{1/2} \\
&\le 2\|\tfrac{d\mathbb{P}}{d\mathbb{Q}}\|_{2,\mathbb{Q}}\,\|f - f_{\boldsymbol{x}}\|_{2,\mathbb{Q}} \\
&\le 2\|\tfrac{d\mathbb{P}}{d\mathbb{Q}}\|_{2,\mathbb{Q}}\Big(\underbrace{\|f - f_\lambda\|_{2,\mathbb{Q}}}_{\text{approximation error}} + \underbrace{\|f_\lambda - f_{\boldsymbol{x}}\|_{2,\mathbb{Q}}}_{\text{sample error}}\Big)
\end{aligned}$$

Outline

Hypothesis space

- $(E, \mathcal{E}, \mathbb{Q})$ probability space, e.g., path space $E = \mathbb{R}^{d \times T}$
- Payoff function $f : E \to \mathbb{R}$, $f \in L^2_{\mathbb{Q}}$
- **Goal:** approximate f by some f_λ in a **hypothesis space** $\mathcal{H}$
- Convenient choice of $\mathcal{H}$: a reproducing kernel Hilbert space ...

Reproducing kernel Hilbert space

Definition 2.1.

A function $k : E \times E \to \mathbb{R}$ is a **kernel** if for any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$ the $n \times n$-matrix $k(x_i, x_j)$ is symmetric and positive semidefinite.

Definition 2.2.

A Hilbert space $\mathcal{H}$ of functions $h : E \to \mathbb{R}$ is a **reproducing kernel Hilbert space (RKHS)** if, for any $x \in E$ there exists $k_x \in \mathcal{H}$ such that

$$\langle h, k_x \rangle_{\mathcal{H}} = h(x), \quad h \in \mathcal{H}.$$

The kernel $k(x, y) = \langle k_x, k_y \rangle_{\mathcal{H}}$ is the **reproducing kernel** of $\mathcal{H}$.

Theorem 2.3 (Moore).

For any kernel $k : E \times E \to \mathbb{R}$ there exists a unique RKHS $\mathcal{H}$ such that $k(\cdot, x) \in \mathcal{H}$ and $\langle h, k(\cdot, x) \rangle_{\mathcal{H}} = h(x)$ for all $h \in \mathcal{H}$ and $x \in E$.

Basic assumptions

- Let $k : E \times E \to \mathbb{R}$ be a measurable kernel with a separable RKHS $\mathcal{H}$
- Then $\mathcal{H}$ consists of measurable functions
- Define $\kappa(x) = \sqrt{k(x,x)} = \|k(\cdot,x)\|_{\mathcal{H}}$, so that

$$|h(x)| \leq \kappa(x)\|h\|_{\mathcal{H}}, \quad h \in \mathcal{H}$$

- **Assume:** $\|\kappa\|_{2,\mathbb{Q}} < \infty$
- Define embedding $J : \mathcal{H} \to L^2_{\mathbb{Q}}$ by $Jh = \mathbb{Q}$-equivalence class of h

Approximation

Approximate f by f_λ in $L^2_{\mathbb{Q}}$, where $h = f_\lambda \in \mathcal{H}$ solves

$$\min_{h\in\mathcal{H}}(\|f-h\|^2_{2,\mathbb{Q}} + \lambda\|h\|^2_{\mathcal{H}}), \tag{2.1}$$

for some regularization parameter $\lambda > 0$:

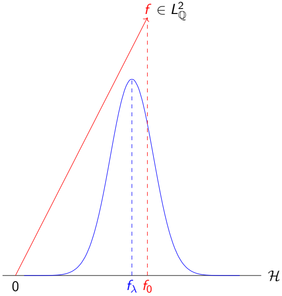


Figure: f_λ with sample error distribution.

Existence, uniqueness, and representer theorem

1. There exists a unique solution $h = f_\lambda \in \mathcal{H}$ to (2.1), and it is given by
$$f_\lambda = (J^*J + \lambda)^{-1} J^* f.$$
2. **[Representer theorem]** The solution f_λ can be represented as
$$f_\lambda = J^* g_\lambda = \int_E k(\cdot, x) g_\lambda(x) \mathbb{Q}(dx)$$
where
$$g_\lambda = (JJ^* + \lambda)^{-1} f.$$

Corollary of representer theorem

Definition 2.4.

We call the kernel k **tractable** if the conditional kernel embeddings

$$M_t(y) = \mathbb{E}_t^{\mathbb{Q}}[k(X, y)]$$

are given in closed form, for all y, t.

Lemma 2.5.

Assume that k is tractable. Then

$$\mathbb{E}_t^{\mathbb{Q}}[f_\lambda(X)] = \int_E M_t(y) g_\lambda(y) \mathbb{Q}(dy)$$

is given in closed form, subject to $\mathbb{Q}$-integration, for all t.

Projection and universal kernels

- Let $f_0 = \mathrm{Proj}_{\overline{\mathrm{Im}\, J}} f$ be the orthogonal projection of f onto $\overline{\mathrm{Im}\, J}$ in $L^2_{\mathbb{Q}}$
- Orthogonality: decompose the squared **approximation error**

$$\|f - f_\lambda\|^2_{2,\mathbb{Q}} = \underbrace{\|f - f_0\|^2_{2,\mathbb{Q}}}_{\text{"best approximation error"}} + \underbrace{\|f_0 - f_\lambda\|^2_{2,\mathbb{Q}}}_{\text{"regularization error"}}.$$

Lemma 2.6.

$\|f_0 - f_\lambda\|_{2,\mathbb{Q}} \to 0$ *as* $\lambda \to 0$ *(but can be slow).*

- Desirable kernel property is $\|f - f_0\|^2_{2,\mathbb{Q}} = 0$:

Definition 2.7.

The kernel k is called **$L^2_{\mathbb{Q}}$-universal** if $\overline{\mathrm{Im}\, J} = L^2_{\mathbb{Q}}$.

Outline

Random sample

- Let $\boldsymbol{x} = (x(1), \ldots, x(n))$ be i.i.d. E-valued rv's with $x(i) \sim \mathbb{Q}$
- Define the empirical measure $\mathbb{Q}_{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^n \delta_{x(i)}$
- All results above apply for $\mathbb{Q}_{\boldsymbol{x}}$ in lieu of $\mathbb{Q}$
- Denote by $J_{\boldsymbol{x}} : \mathcal{H} \to L^2_{\mathbb{Q}_{\boldsymbol{x}}}$ the sample version of $J : \mathcal{H} \to L^2_{\mathbb{Q}}$
- Obtain $\mathcal{H}$-valued sample estimator

$$f_{\boldsymbol{x}} = (J_{\boldsymbol{x}}^* J_{\boldsymbol{x}} + \lambda)^{-1} J_{\boldsymbol{x}}^* f = J_{\boldsymbol{x}}^* \underbrace{(J_{\boldsymbol{x}} J_{\boldsymbol{x}}^* + \lambda)^{-1} f}_{= g_{\boldsymbol{x}} \in L^2_{\mathbb{Q}_{\boldsymbol{x}}}}$$

How to compute $f_{\boldsymbol{x}}$

- Denote $\boldsymbol{f} = (f(x(1)), \dots, f(x(n)))^\top$
- Define $n \times n$-kernel matrix $\boldsymbol{K}_{ij} = k(x(i), x(j))$
- $\frac{1}{n}\boldsymbol{K}$ is matrix representation of $J_{\boldsymbol{x}} J_{\boldsymbol{x}}^* : L^2_{\mathbb{Q}_x} \to L^2_{\mathbb{Q}_x}$

Corollary 3.1 (of representer theorem).

The unique solution $\boldsymbol{g} \in \mathbb{R}^n$ *to*

$$(\tfrac{1}{n}\boldsymbol{K} + \lambda)\boldsymbol{g} = \boldsymbol{f}$$

gives $f_{\boldsymbol{x}} = \frac{1}{n}\sum_{j=1}^n k(\cdot, x(j))\boldsymbol{g}_j$*. If, moreover, the kernel* k *is tractable then*

$$V_{\boldsymbol{x},t} = \frac{1}{n}\sum_{j=1}^n M_t(x(j))\boldsymbol{g}_j$$

is given in closed form.

Law of large numbers and central limit theorem

Theorem 3.2.

Assume that $\|\kappa\|_{4,\mathbb{Q}} < \infty$ *and* $\|f\kappa\|_{2,\mathbb{Q}} < \infty$.

1. *Law of large numbers:* $f_{\boldsymbol{x}} \xrightarrow{a.s.} f_\lambda$ *as* $n \to \infty$
2. *Central limit theorem:*

$$\sqrt{n}(f_{\boldsymbol{x}} - f_\lambda) \xrightarrow{d} \mathcal{N}(0, Q) \quad \text{as } n \to \infty$$

with covariance operator $Q : \mathcal{H} \to \mathcal{H}$ *given by*

$$\langle Qh, h\rangle_{\mathcal{H}} = \mathbb{V}_{\mathbb{Q}}[(f - f_\lambda)(J^*J + \lambda)^{-1}h], \quad h \in \mathcal{H}.$$

Finite sample guarantee

Theorem 3.3.

Assume that

$$\|f\|_{\infty,\mathbb{Q}} < \infty \quad \text{and} \quad \|\kappa\|_{\infty,\mathbb{Q}} < \infty. \tag{3.1}$$

For any $\eta \in (0,1]$, *we have*

$$\|f_{\boldsymbol{x}} - f_\lambda\|_{\mathcal{H}} < \frac{2\sqrt{2\log(2/\eta)}\|(f - f_\lambda)\kappa\|_{\infty,\mathbb{Q}}}{\lambda\sqrt{n}}$$

with sampling probability of at least $1-\eta$.

Problem: Assumption (3.1) does not hold in finance in general!

Weighted sampling

- Equivalent sampling measure $\widetilde{\mathbb{Q}} \sim \mathbb{Q}$ with RN derivative $w = \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}$
- Induces isometric isomorphism

$$\begin{array}{ccc} \widetilde{\mathcal{H}} & \xrightarrow{\widetilde{J}} & L^2_{\widetilde{\mathbb{Q}}} \\ {\scriptstyle \times\sqrt{w}}\Big\downarrow & & \Big\uparrow{\scriptstyle \times\frac{1}{\sqrt{w}}} \\ \mathcal{H} & \xrightarrow{J} & L^2_{\mathbb{Q}} \end{array}$$

$$\widetilde{f} = \frac{f}{\sqrt{w}} \text{ and } \widetilde{k}(x, y) = \frac{k(x,y)}{\sqrt{w(x)w(y)}}$$

Solution:

1. Choose $\widetilde{\mathbb{Q}}$ such that $\|\widetilde{f}\|_{\infty,\mathbb{Q}} < \infty$ and $\|\widetilde{\kappa}\|_{\infty,\mathbb{Q}} < \infty$.
2. Learn $\widetilde{f}_{\boldsymbol{x}}$ in $\widetilde{\mathcal{H}}$ and **define** $f_{\boldsymbol{x}} := \widetilde{f}_{\boldsymbol{x}} \times \sqrt{w}$
3. Obtain finite sample guarantee extending Theorem 3.3

$$\|f_{\boldsymbol{x}} - f_\lambda\|_{\mathcal{H}} < \frac{2\sqrt{2\log(2/\eta)}\|\frac{1}{w}(f - f_\lambda)\kappa\|_{\infty,\mathbb{Q}}}{\lambda\sqrt{n}}$$

Particular choice of sampling measure

Optimal sampling measure that minimizes $\|\widetilde{\kappa}\|_{\infty,\mathbb{Q}}$:

Lemma 3.4.

For any $\widetilde{\mathbb{Q}} \sim \mathbb{Q}$, we have $\|\widetilde{\kappa}\|_{\infty,\mathbb{Q}} \geq \|\kappa\|_{2,\mathbb{Q}}$, with equality if and only if

$$w = \frac{\kappa^2}{\|\kappa\|_{2,\mathbb{Q}}^2}, \quad \mathbb{Q}\text{-a.s.}$$

In this case, $\widetilde{\kappa} = \|\kappa\|_{2,\mathbb{Q}}$ is constant $\mathbb{Q}$-a.s.

Solution: Choose tractable kernel k such that $\frac{f}{\kappa}$ is bounded

Outline

Product structure

- Random driver $X = (X_1, \dots, X_T)$, e.g., Brownian motion
- Path space $E = E_1 \times \cdots \times E_T$, e.g., $E = \mathbb{R}^{d \times T}$
- Product measure $\mathbb{Q}(dx) = \mathbb{Q}_1(dx_1) \times \cdots \times \mathbb{Q}_T(dx_T)$
- Product kernel $k(x, y) = k_1(x_1, y_1) \times \cdots \times k_T(x_T, y_T)$
- RKHS tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_T$

Tractable kernels

- Assume for all t: k_t is tractable, i.e., kernel embeddings

$$m_t(y) = \int_{E_t} k_t(x, y)\mathbb{Q}_t(dx)$$

  in closed form

- Then k is tractable: conditional kernel embeddings given by

$$M_t(y) = \mathbb{E}_t^{\mathbb{Q}}[k(X, y)] = \prod_{s=1}^{t} k_s(X_s, y_s) \prod_{s=t+1}^{T} m_s(y_s)$$

Lévy models ($E_t \subseteq \mathbb{R}^d$)

- Bocher's theorem: for symmetric p.m. Λ on $\mathbb{R}^d$, $\beta \geq 0$, obtain kernel

$$k_t(x,y) = \mathrm{e}^{\beta x^\top y} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(x-y)^\top \lambda} \Lambda(d\lambda)$$

- Lévy driver: if $\mathbb{Q}_t$ infinitely divisible then kernel embedding

$$m_t(x) = \int_{\mathbb{R}^d} \widehat{\mathbb{Q}}_t(\beta x + \mathrm{i}\lambda) \mathrm{e}^{-\mathrm{i}x^\top \lambda} \Lambda(d\lambda)$$

  is in closed form subject to Λ-integration

- Lévy kernels: if Λ is infinitely divisible then kernel is in closed form

$$k_t(x,y) = \mathrm{e}^{\beta x^\top y} \mathrm{e}^{-\frac{1}{2}(x-y)^\top A(x-y) + \int_{\mathbb{R}^d} (\cos((x-y)^\top \xi) - 1)\nu(d\xi)}$$

- Example: **Gaussian-exponentiated kernel**, $\alpha \geq 0$,

$$k_t(x,y) = \mathrm{e}^{\beta x^\top y - \alpha \|x-y\|^2}$$

Outline

Gaussian white noise $\mathbb{Q} = N(0, I_{d\times T})$

Gaussian-exponentiated kernel, for $\alpha \geq 0$, $\beta \in [0, 1/2)$,

$$k(x, y) = \mathrm{e}^{-\alpha\|x-y\|^2+\beta x^\top y}$$

with RN derivative, for $\gamma < 1/2$,

$$w(x) = (1-2\gamma)^{dT/2}\mathrm{e}^{\gamma\|x\|^2}$$

Then:

- kernel is tractable: $m_s(y_s) = (1+2\alpha)^{-d/2}\mathrm{e}^{\frac{\beta^2+4\alpha\beta-2\alpha}{4\alpha+2}\|y_s\|^2}$
- sampling measure is normal: $\widetilde{\mathbb{Q}} = N(0, \frac{1}{1-2\gamma} I_{d\times T})$
- $\widetilde{k}$ is $L^2_{\widetilde{\mathbb{Q}}}$-universal: $\overline{\mathrm{Im}\,\widetilde{J}} = L^2_{\widetilde{\mathbb{Q}}}$ and hence $f_0 = f$
- $\|\widetilde{\kappa}\|_{\infty,\mathbb{Q}} < \infty \Leftrightarrow \beta \leq \gamma$ (Lemma 3.4: $\beta = \gamma$ optimal)

Black–Scholes model

- T time steps at length Δ_t [years], $t = 1, \ldots, T$
- $X = (X_1, \cdots, X_T) \sim N(0, I_{d \times T})$ Gaussian white noise
- Black–Scholes discrete time model with d nominal price processes

$$S_{i,t} = S_{i,t-1} \mathrm{e}^{(r+\sigma_i^2/2)\Delta_t + \sigma_i \sqrt{\Delta_t} X_{i,t}}, \quad S_{i,0} = 1$$

- Volatility parameters $\sigma_i = 0.2$
- Risk-free rate $r = 0$

European max-call and min-put options

- $T = 2$ time steps at length $\Delta_1 = 1/12$ and $\Delta_2 = 11/12$ [years]
- Max-call option has discounted payoff at T

$$f(X) = \mathrm{e}^{-rT}(\max_i S_{i,T} - K)^+$$

- Min-put option has discounted payoff at T

$$f(X) = \mathrm{e}^{-rT}(K - \min_i S_{i,T})^+$$

- Strike price $K = S_{i,0} = 1$
- Number of stocks $d = 6$
- **Total dimension** $d \times T = 12$

Barrier reverse convertible

- $T = 12$ time steps at length $\Delta_t = 1/12$ [years]
- Barrier reverse convertible has discounted payoff at T

$$f(X) = \mathrm{e}^{-rT} C + \mathrm{e}^{-rT} F \left(1 - 1_{\{\min_{i,t} S_{i,t} \le B\}} \left(1 - \min_i \frac{S_{i,T}}{S_{i,0} K} \right)^+ \right)$$

- Face value $F = 1$
- Coupon $C = 0$
- Barrier $B = 0.6$
- Strike price $K = S_{i,0} = 1$
- Number of underlying prices $d = 3$
- **Total dimension** $d \times T = 36$

Numerical results

- Training sample size n varies between $4,000$ and $20,000$
- Hyperparameters α, β, λ via GPR
- Compare true and estimated value process

$$V_t = \mathbb{E}_t^{\mathbb{Q}}[f(X)], \quad V_{\mathbf{x},t} = \mathbb{E}_t^{\mathbb{Q}}[f_{\mathbf{x}}(X)]$$

- Ground truth: nested MC with 10^5 outer and 10^3 inner simulations

Below we report

- normalized $L^2_{\mathbb{Q}}$-errors $\frac{\|V_t - V_{\mathbf{x},t}\|_{2,\mathbb{Q}}}{V_0}$ in % for $t = 0, 1, T$
- detrended Q-Q plots of $V_{\mathbf{x},t}$ against V_t for $t = 1, T$

Optimal hyperparameter values α, β, λ from GPR

Payoff	α	β	λ
Min-put	2.06×10^{-2}	0	1.86×10^{-8}
Max-call ($\gamma = 0$)	2.53×10^{-2}	0	3.33×10^{-8}
Max-call ($\gamma = 0.15$)	2.66×10^{-2}	4.10×10^{-9}	4.14×10^{-8}
Barrier reverse convertible	2.95×10^{-3}	0	9.19×10^{-8}

- Optimal values can differ across portfolios

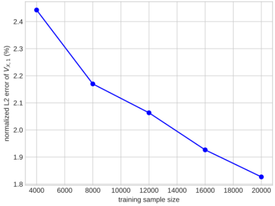
Normalized $L^2_{\mathbb{Q}}$-errors $\|V_t - V_{\mathbf{x},t}\|_{2,\mathbb{Q}}/V_0$ in %

Payoff	$t = 0$	1	T
Min-put	0.194	1.83	10.1
Max-call ($\gamma = 0$)	0.080	2.50	12.4
Max-call ($\gamma = 0.15$)	0.103	2.32	11.7
Barrier reverse convertible	0.022	0.25	5.8

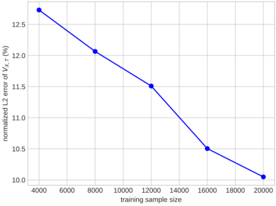
- In line with basic estimation error bound

$$\|V_t - V_{\mathbf{x},t}\|_{2,\mathbb{Q}} \leq \underbrace{\|f - f_{\mathbf{x}}\|_{2,\mathbb{Q}}}_{\text{estimate by MC}}$$

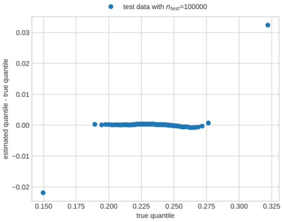
Results for the min-put



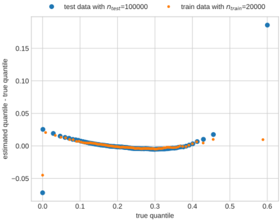
Normalized $L^2_{\mathbb{Q}}$-error of $V_{\boldsymbol{x},1}$



Normalized $L^2_{\mathbb{Q}}$-error of $V_{\boldsymbol{x},T}$

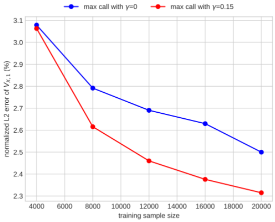


Detrended Q-Q plot of $V_{\boldsymbol{x},1}$

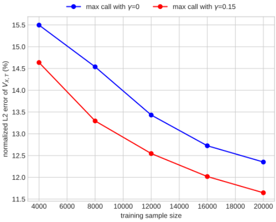


Detrended Q-Q plot of $V_{\boldsymbol{x},T}$

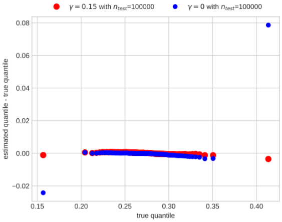
Results for the max-call with $\gamma = 0$ and $\gamma = 0.15$



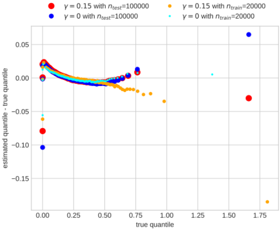
Normalized $L^2_{\mathbb{Q}}$-error of $V_{\boldsymbol{x},1}$



Normalized $L^2_{\mathbb{Q}}$-error of $V_{\boldsymbol{x},T}$

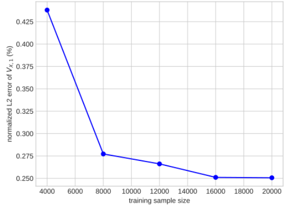


Detrended Q-Q plots of $V_{\boldsymbol{x},1}$

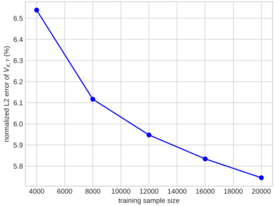


Detrended Q-Q plots of $V_{\boldsymbol{x},T}$

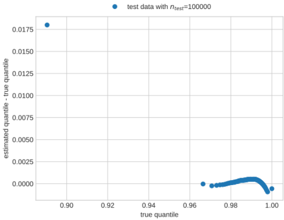
Results for the barrier reverse convertible



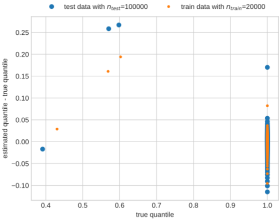
Normalized $L^2_{\mathbb{Q}}$-error of $V_{\boldsymbol{x},1}$



Normalized $L^2_{\mathbb{Q}}$-error of $V_{\boldsymbol{x},T}$



Detrended Q-Q plot of $V_{\boldsymbol{x},1}$



Detrended Q-Q plot of $V_{\boldsymbol{x},T}$

Outline

Regress-now

- Fix a risk horizon, e.g., $t = 1$
- Directly learn $V_1 = \mathbb{E}_1^{\mathbb{Q}}[f(X)]$ from "noisy" observations

$$f(X) = V_1 + \underbrace{f(X) - V_1}_{\text{noise } \epsilon}$$

- Straightforward adaptation: restrict to RKHS $\mathcal{H}_1$ with kernel k_1 on E_1
- Leads to regress-now estimator $V_{\boldsymbol{x},1}^{\text{now}} = \frac{1}{n}\sum_{j=1}^{n} k_1(X_1, x(j))\boldsymbol{g}_j^{\text{now}}$
- Advantage: smaller state space $\dim E_1 < \dim(E_1 \times \cdots \times E_T)$
- But: dimension of regression problem remains the same, $n = 20,000$
- **Result:** regress-later performs better than regress-now, both in terms of accuracy and computing times!

Computing times

Payoff	regress-now	regress-later
Min-put	3122	2756
Max-call	3288	2570
Barrier reverse convertible	3683	2756

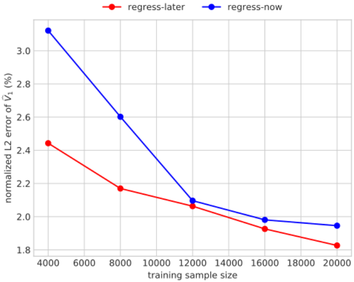
Table: Computing times (in seconds) for estimating $V_{x,1}^{\text{now}}$ and $V_{x,1}$, for sample size $n = 20,000$, on Skylake processors running at 2.3 GHz, using 14 cores and 100 GB of RAM.

Normalized $L^2_{\mathbb{Q}}$-errors $\|V_1 - \widehat{V}_1\|_{2,\mathbb{Q}}/V_0$ in %

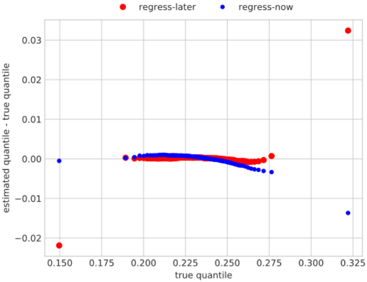
Payoff	regress-now	regress-later
Min-put	1.95	1.83
Max-call	2.61	2.50
Barrier reverse convertible	0.281	0.25

Table: Normalized $L^2_{\mathbb{Q}}$-error $\|V_1 - \widehat{V}_1\|_{2,\mathbb{Q}}/V_0$ in % for $\widehat{V}_1 \in \{V^{\text{now}}_{x,1}, V_{x,1}\}$

Results for the min-put

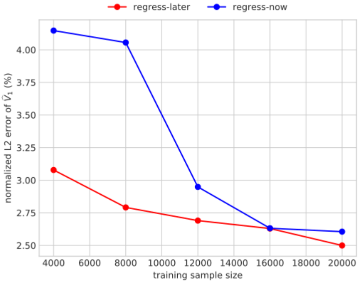


normalized $L^2_{\mathbb{Q}}$-errors of $\widehat{V}_1$ in %

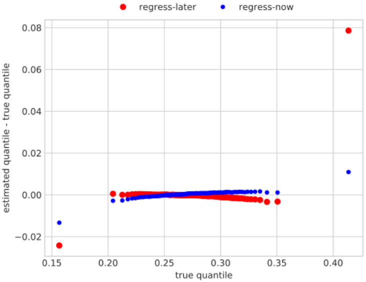


detrended Q-Q plots of $\widehat{V}_1$

Results for the max-call (with $\gamma = 0$)

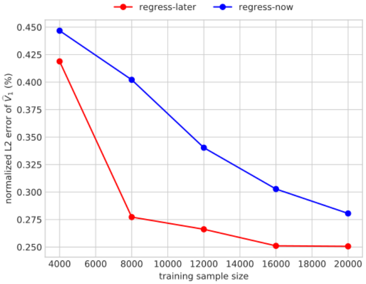


normalized $L^2_{\mathbb{Q}}$-errors of $\widehat{V}_1$ in %

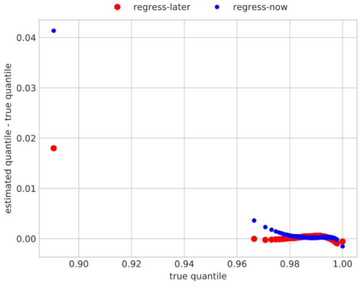


detrended Q-Q plots of $\widehat{V}_1$

Results for the barrier reverse convertible



normalized $L^2_{\mathbb{Q}}$-errors of $\widehat{V}_1$ in %



detrended Q-Q plots of $\widehat{V}_1$

Conclusion

- Computational framework for quantitative portfolio risk management
- Machine learning with kernels gives accurate estimates of the dynamic value process for relatively small training sample size
- Weighted sampling further improves results (finite sample guarantees)
- Regress-later outperforms regress-now
- Scalability limits at dimensions beyond $d \times T = 100$