Backward propagation of chaos and large population games asymptotics

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$$X_t^{i,n} = \xi^i + \int_0^t b_u\Big(X_u^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_u^{j,n}}\Big)\, du + \int_0^t \sigma_u\Big(X_u^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_u^{j,n}}\Big)\, dW_u^i$$

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$$X^{i,n} \to X^i$$

with

$$X_t^i = \xi^i + \int_0^t b_u(X_u^i, \mathrm{law}(X_u^i))\,du + \int_0^t \sigma_u(X_u^i, \mathrm{law}(X_u^i))\,dW_u^i,$$

$\rightsquigarrow$ McKean-Vlasov SDE.

- Propagation of chaos: $\text{law}(X^{1,n}, \dots, X^{k,n}) \to \text{law}(X^i)^{\otimes k}$
- Concentration: $P\Big(\mathcal{W}_2(\frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,n}}, \text{law}(X_t^i)) \geq x\Big) \leq 2e^{-Cnx^2}$
- Approximation and trend to equilibrium for nonlocal Fokker-Plank equations, e.g. $\partial_t \rho = \nabla \cdot (\nabla \rho + \rho \nabla V * \rho)$

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How about backward particles?

Backward propagation of chaos

For $\boldsymbol{x} = (x_1, \ldots, x_n) \in (\mathbb{R}^m)^n$, put

$$L^n(\boldsymbol{x}) := \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i}.$$

- Let $(G^1, \ldots, G^n)$ be i.i.d. $\mathcal{F}_T^n$-measurable and consider the system

$$Y_t^{i,n} = G^i + \int_t^T F_u(Y_u^{i,n}, Z_u^{i,i,n}, L^n(\boldsymbol{Y}_u))\, du - \sum_{k=1}^{n} \int_t^T Z_u^{i,k,n}\, dW_u^k$$

where $(W^i, \ldots, W^n)$ are n independent Brownian motions.

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  where $(W^i, \ldots, W^n)$ are n independent Brownian motions.
- Further consider the McKean-Vlasov BSDE

$$Y_t^i = G^i + \int_t^T F_u(Y_u^i, Z_u^i, \mathcal{L}(Y_u))\, du - \int_t^T Z_u^i\, dW_u^i.$$

Theorem

If $F : [0,T] \times \mathcal{C} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathcal{P}_2(\mathbb{R}^m) \to \mathbb{R}^m$ *is Lipschitz continuous (in* (y, z, μ)*) and there is* $k > 2$ *such that* $E[|G|^k] < \infty$*, then*

$$\sup_t E\left[\mathcal{W}_2^2(L^n(\boldsymbol{Y}_t), \mathcal{L}(Y_t))\right] \leq C r_{n,m,k}$$

for some rate $r_{n,m,k} \downarrow 0$ *as* $n \to \infty$*.*

If $E[|G|^k] < \infty$ *for some* $k > 4$*, then*

$$P\Big(\mathcal{W}_2(L^n(\boldsymbol{Y}_t), \mathcal{L}(Y_t)) \geq x\Big) \leq C\tilde{r}_{n,x,k}$$

with $\tilde{r}_{n,x,k} \downarrow 0$ *(as* $n \to \infty$*) exponentially fast.*

$r_{n,m,k}$ and $\tilde{r}_{n,x,k}$ are explicitly given, and depends on m and k.

Lemma

Lipschitz continuity yields

$$\mathcal{W}_2(L^n(\boldsymbol{Y}_t), \mathcal{L}(Y_t)) \leq e^{L_F T} \mathcal{W}_2(L^n(\tilde{\boldsymbol{Y}}_t), \mathcal{L}(Y_t))$$

where $\tilde{\boldsymbol{Y}} := (\tilde{Y}^1, \ldots, \tilde{Y}^n)$ *and* $(\tilde{Y}^1, \tilde{Z}^1), \ldots (\tilde{Y}^n, \tilde{Z}^n)$ *are iid copies of* (Y, Z) *solving*

$$\tilde{Y}^i_t = G^i + \int_t^T F_u(\tilde{Y}^i_u, \tilde{Z}^i_u, \mathcal{L}(Y_u))\, du - \int_t^T \tilde{Z}^i_u\, dW^i_u.$$

- By Bartl & T. (2020), $\mathcal{L}(Y)$ satisfies dimension-free Talagrand inequality.
- Use results by Fournier & Guillin (2015) and Horowitz & Karandikar (1994) to conclude.
- See also Sznitman.

- Crowd motion with tagged pedestrians $\rightsquigarrow$ Aurell & Djehiche (2018, 2019)

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  - Relaxed controls
$$\alpha \curvearrowright q \in \mathcal{P}([0,T] \times \mathbb{A})$$
$\rightsquigarrow$ Lacker (2016, 2018); Fischer (2017); Djete (2020)

  - PDE characterizations (PDE on the Wasserstein space)
$$\alpha_t \equiv \alpha(t, X_t)$$
$\rightsquigarrow$ Delarue, Lacker, Raman (2018); Cardaliaguet, Delarue, Lasry, Lions (2019).

Large population games asymptotics:
The Markovian case

- Consider a large number of investors trading a stock S. Their inventories is

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- Assuming trading has a permanent impact on the stock price S, we have (c.f. Almgren-Chriss 00')

$$dS_t = \frac{1}{n} \sum_{j=1}^{n} \alpha_t^j \, dt + \sigma^o dW_t^o$$

- Assuming (quadratic) transaction costs $\frac{1}{2}|\alpha_t^i|^2$, the wealth of trader i is

$$dV_t^i = \Big(-\frac{1}{2}|\alpha_t^i|^2 + X_t^i \frac{1}{n} \sum_{j=1}^{n} \alpha_t^j \Big) \, dt + X_t^i \sigma \, dW_t^o + S_t \sigma \, dW_t^i .$$

- If the traders face the terminal liquidation constrain $g(X_T^i)$ and are risk neutral, then they have to solve

$$J(\underline{\alpha}) = E\Big[V_T^i - g(X_T^i)\Big] \to \max$$

That is,

$$\begin{cases} \inf_{\alpha^i} E\Big[\int_0^T \frac{1}{2}|\alpha_t^i|^2 - X_t^i \frac{1}{n}\sum_{j=1}^n \alpha_t^j \, dt + g(X_T^i)\Big] \\ dX_t^i = \alpha_t^i \, dt + \sigma \, dW_t^i \end{cases}$$

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- As usual, one is interested in computing a Nash equilibrium.

- The mean field game analogue of this game takes the following form: Fix a flow of measure $(\nu_t)_{t\in[0,T]}$ and find $\hat{\alpha}^{\nu}$ solving

$$\begin{cases} \inf_{\alpha} E\Big[\int_0^T \frac{1}{2}|\alpha_t|^2 - X_t \int x\nu_t(dx)\,dt + g(X_T^{\alpha})\Big] \\ dX_t^{\alpha} = \alpha_t\,dt + \sigma\,dW_t^i. \end{cases}$$

- The mean field equilibrium is $\hat{\alpha} = \hat{\alpha}^{\hat{\nu}}$ such that $\mathrm{law}(\hat{\alpha}_t^{\hat{\nu}}) = \hat{\nu}_t$.
- It is the optimal strategy for an infinitesimal representative player

$\leadsto$ Lasry, Lions 2006; Huang, Malhamé, and Caines 2006. Gomes, Patrizi, and Voskanyan 2014

Theorem

If $\hat{\boldsymbol{\alpha}}^n = (\hat{\alpha}^{i,n}, \ldots, \hat{\alpha}^{i,n})$ *is a Nash equilibrium, then it satisfies* $\alpha_t^{i,n} = -\frac{1}{1+\frac{1}{n}X_t^{i,n}} Y_t^{ii,n}$ *with* $(X^{i,n}, Y^{ij,n}, Z^{ijk,n})$ *solving the FBSDE*

$$\begin{cases} dX_t^{i,n} = -\frac{1}{1+\frac{1}{n}X_t^{i,n}} Y_t^{ii,n}\, dt + \sigma\, dW_t^i \\ dY_t^{ij,n} = -\delta_{ij} \frac{1}{n} \sum_{k=1}^n \hat{\alpha}_t^{k,n}\, dt - \sum_{k=1}^n Z_t^{ijk,n}\, dW_t^k \\ Y_T^{ij,n} = \delta_{ij} g'(X_T^{i,n}) \end{cases}$$

Theorem

Assume that $E[|X_0^i|^k] < \infty$ for $k > 2$ and g Lipschitz continuous. If $\hat{\boldsymbol{\alpha}}^n = (\hat{\alpha}^{i,n}, \dots, \hat{\alpha}^{i,n})$ is a Nash equilibrium, then there is $\hat{\alpha}^i$ such that for all $i = 1, \dots, n$ it holds

$$E[|\hat{\alpha}_t^{i,n} - \hat{\alpha}_t^i|^2] \leq C r_{n,m,d,k}$$

and $\hat{\alpha}^i$ is a mean field equilibrium. In addition, if X_0^i is constant then for n large enough,

$$P(\mathcal{W}_2(L^n(\hat{\boldsymbol{\alpha}}_t^n), \mathcal{L}(\hat{\alpha}_t^i)) \geq a) \leq \frac{C}{a^2 n^2} + C e^{-a^2 n^2}.$$

Large population games asymptotics:
The non-Markovian case

- If the traders face the terminal liquidation constrain $g(X^i, L^n(\boldsymbol{X}))$ and are risk neutral, then they have to solve

$$\begin{cases} \inf_{\alpha^i} E\Big[\int_0^T \frac{1}{2}|\alpha_t^i|^2 - X_t^i \frac{1}{N}\sum_{j=1}^N \alpha_t^j \, dt + g(X^i, L^n(\boldsymbol{X}))\Big] \\ dX_t^i = \alpha_t^i \, dt + \sigma \, dW_t^i \end{cases}$$

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- We rather consider the weak formulation of the problem:

$$\begin{cases} \inf_{\alpha^i} E^{P^{\boldsymbol{\alpha}}}\Big[\int_0^T \frac{1}{2}|\alpha_t^i|^2 - X_t^i \frac{1}{N}\sum_{j=1}^N \alpha_t^j\,dt + g(X^i, L^n(\boldsymbol{X}))\Big] \\ dX_t^i = \alpha_t^i\,dt + \sigma\,dW_t^i \\ dP^{\boldsymbol{\alpha}} = \mathcal{E}(\sum_{i=1}^n \int \alpha_t^i\,dW_t^i)dP \end{cases}$$

- The mean field game analogue of this game takes the following form: Fix a flow of measure $\xi_t = (\mu_t, \nu_t)$ and find $\hat{\alpha}^{\xi}$ solving

$$\begin{cases} \inf_{\alpha} E^{P^{\alpha}}\Big[\int_0^T \frac{1}{2}|\alpha_t|^2 - X_t \int x\nu_t(dx) + g(X,\mu)\Big] \\ dX_t^i = \alpha_t\, dt + \sigma\, dW_t^i \\ dP^{\alpha} = \mathcal{E}(\int_0^T \alpha\, dW_t) dP. \end{cases}$$

- The mean field equilibrium is $\hat{\alpha} = \hat{\alpha}^{\xi}$ such that $P^{\hat{\alpha}} \circ (X_t, \hat{\alpha}_t^{\xi})^{-1} = \xi_t$.

$\rightsquigarrow$ Carmona & Lacker 2015

Theorem

If $\hat{\alpha}^n$ is a Nash equilibrium, then it satisfies $\hat{\alpha}^{i,n}_t = \frac{1}{1+\frac{1}{n}X^i_t} Z^{ii,n}_t$ where $(Y^{i,n}, Z^{ij,n})$ solves

$$\begin{cases} dY^{i,n}_t = -\left\{ f(X^i_t/n, Z^{ii,n}) + X_t \frac{1}{n} \sum_{j=1}^n Z^{jj,n}_t \right\} dt - \sum_{j=1}^n Z^{ij,n}\, dW^j_t \\ Y^{i,n}_T = g(X^i, L^n(\boldsymbol{X})) \end{cases}$$

Theorem

Assume that g is bounded and Lipschitz continuous. If there is a Nash equilibrium $\hat{\alpha}^N$, *then there is* $\hat{\alpha}^i$ *such that for all i it holds*

$$E^{P^{\hat{\alpha}^i}}\Big[\int_0^T |\hat{\alpha}_t^{i,n} - \hat{\alpha}_t^i|^2\,dt\Big] \le C E^{P^{\hat{\alpha}^i}}\big[\mathcal{W}_{2,L^2}(L^n(\boldsymbol{X}), \mathcal{L}_{\hat{\alpha}}(X))\big] + C r_n.$$

and $\hat{\alpha}^i$ *is a mean-field equilibrium. In particular, up to a subsequence,* $(\hat{\alpha}_t^{i,n})$ *converges to* $\hat{\alpha}_t^i$ $P^{\hat{\alpha}^i} \otimes dt$*-a.s.*

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  - Convergence rates and concentration inequalities
  - Non uniqueness assumed
  - The Markovian case has interesting PDE interpretations.
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  - "Explicit" representations of equilibria
- Some issues
  - Explicit representations needed
  - Closed-loop controls
  - Symmetry essential

Summary

- Backward propagation of chaos
  - explicit convergence rates
  - concentration inequalities
- Forward backward "particles"
  - convergence to extended MFG
- Interaction through control process
  - convergence to non-Markovian extended MFG

Thank You!