Optimal fund menus

Jaksa Cvitanic
Caltech

Julien Hugonnier
EPFL
Motivation and Preview

- Why do asset managers offer so many different mutual funds?
- What is the optimal menu of funds?
- This paper provides a novel model:
  - The characteristic(s) of investors are unobserved.
  - Fees are linear within a given class of mutual fund shares.
  - Bundling—i.e. offering funds rather than the underlying assets—is the optimal way to screen investors.
- This leads to a new class of screening problems that are subject to a linear pricing constraint.
- Main results:
  - an alternative explanation for "closet indexing" or "home bias".
  - linear pricing constraint increases investors' welfare
The model

• Static model with three assets:
  • A riskless asset with gross return $r$
  • Two risky assets whose gross excess returns $\epsilon$ are independent random variables with unit variance.

• One manager and unit mesure of risk-averse investors.

• Investors of type $\theta \in \Theta = [0, \theta_H]$ believe that $E_\theta[\epsilon] = \xi(\theta) = (\xi, \theta)^\top$ and their preferences are given by

$$u(\theta; w_1) = aE_\theta[w_1 - rw_0] - \frac{a^2}{2}\text{var}_\theta[w_1]$$

where $w_0 = 1$ and $a = 1$ are constants that represent the common initial wealth and risk aversion of investors.
The model

- Investors can only access the risky assets by investing in the funds offered by the manager (and paying the cost).

Q: What should the manager offer given that he only knows that types are uniformly distributed across investors?

- If any pricing scheme is allowed this is a standard screening problem for which the optimal strategy is to offer:
  1. A fixed cost for the index (no surplus for investors),
  2. A quantity dependent fee for the non index (excludes $\theta \leq \frac{1}{2} \left| \Theta \right| / 2$).

- But nonlinear pricing is not allowed within a given class of shares.

- We solve for the manager’s optimal offering strategy under a linear pricing constraint.
The model

• A (linearly priced) fund is specified by a pair \((\gamma, \phi) \in \mathbb{R}_+ \times \mathbb{R}^2\).

• A fund menu is a triple \(\mathbf{m} = (\gamma, \phi, \mathcal{M})\) where \(\mathcal{M}\) is an index set and \(\gamma, \phi : \mathcal{M} \to \mathbb{R}_+ \times \mathbb{R}^2\) are functions that describe the funds.

• The best response of an investor of type \(\theta\) to a fund menu \(\mathbf{m}\) offered by the manager is the measure

\[
q^*(\theta, \mathbf{m}) = \arg \max_{q \in \mu_+ (\mathcal{M})} u \left( \theta; r + \int_{\mathcal{M}} (\phi(m)^\top \epsilon - \gamma(m)) q(dm) \right),
\]

and the aggregation of these best responses generates

\[
\int_{\mathcal{M} \times \Theta} \gamma(m) q^*(dm; \theta, \mathbf{m}) \frac{d\theta}{|\Theta|}
\]

in fees for the manager.
The revelation principle

• Denote by

\[ \pi(\theta, \phi) = \arg \max_{q \geq 0} u(\theta; r + q(\phi^\top \epsilon - 1)) \]

the amount that an investor of type \( \theta \) optimally allocates when the manager offers a single fund with characteristics \((1, \phi)\).

• Proposition. Given \( \bar{m} \) there exists \( m \) such that

1. \( \mathcal{M}(m) = \Theta \)
2. \( \gamma(\theta, m) = 1 \) for all \( \theta \in \Theta \)
3. \( q^*(dm; \theta, m) = \pi(\theta, \phi(\theta))\delta_{\{\theta\}}(dm) \) for all \( \theta \in \Theta \)
4. The manager and investors are indifferent between \( \bar{m} \) and \( m \)
Incentive compatibility

• The problem reduces to maximizing

\[ I(\phi) = \int_{\Theta} \pi(\theta, \phi(\theta)) \frac{d\theta}{|\Theta|} \]

over the set of fund loading functions that induce each investor to only allocate money to the fund that targets his type.

• **Proposition.** A fund loading function is **incentive compatible** if and only if it satisfies the inequality

\[ \alpha(\theta' | \theta) := \phi(\theta')^T \xi(\theta) - 1 - \frac{\phi(\theta')^T \phi(\theta)}{\|\phi(\theta)\|^2} (\phi(\theta)^T \xi(\theta) - 1)_+ \leq 0 \]  

(IC)

for all pairs of investor types \((\theta, \theta') \in \Theta\).
1st best benchmark

• Suppose that the manager knows the type of each investor.
• In this case his optimal strategy is to offer investors of type $\theta$ a single fund with unit fee and loadings

$$\phi^*(\theta) = \arg \max_{\phi \in \mathbb{R}^2} \pi(\theta, \phi) = \frac{2\xi(\theta)}{\|\xi(\theta)\|^2}.$$  

• Substituting into (IC) shows that (for this menu):

$$\alpha(\theta' | \theta) = \frac{\theta'(\theta - \theta')}{\|\xi(\theta)\|^2} > 0, \quad \theta' < \theta.$$  

$\Rightarrow$ If the manager was to offer this menu each investor would have an incentive to report a lower type than his own.
Instruments

- The problem reduces to maximizing $I(\phi)$ over the set $\Phi$ of loading functions $\phi \in AC(\Theta; \mathbb{R})$ that satisfy condition (IC).

- This problem is hard because the constraint is non-standard and the optimization is over vector valued functions.

- **Lemma.** If $\phi \in \Phi$ then, with the value function

  $$
  v(\theta) = \max_{q \geq 0} u(\theta; r + q (\phi(\theta)^T \epsilon - 1))
  $$

  we have, from the first order condition, that,

  $$
  \pi(\theta, \phi(\theta)) = \theta \dot{v}(\theta) - 2v(\theta) + \xi \sqrt{2v(\theta) - [\dot{v}(\theta)]^2}
  $$

  for every investor type $\theta \in \Theta$. 

Optimal fund menus
The relaxed problem

- It follows that

$$\sup_{\phi \in \Phi} l(\phi) \leq \sup_v \int_{\Theta} \left( \theta \dot{v}(\theta) - 2v(\theta) + \xi \sqrt{2v(\theta)} - [\dot{v}(\theta)]^2 \right) \frac{d\theta}{|\Theta|}$$

where the **relaxed problem** on the right only takes into account the 1st order condition associated with (IC).

- The Euler-Lagrange equation is, with \(F(\theta,v,v')\) denoting the integrand,

$$0 = F_v(\theta) - \frac{d}{d\theta} F_{\dot{v}}(\theta), \quad \text{on } \Theta, \quad \text{(EL)}$$

$$0 = F_{\dot{v}}(\theta), \quad \text{on } \partial \Theta. \quad \text{(BC)}$$

Because the integrand is concave in \((v, \dot{v})\) this Boundary Value Problem is necessary and sufficient for optimality in the relaxed problem.
The optimal fund menu

- **Main results.**
  1. (EL) and (BC) admit a unique solution $v^* \in C^2(\Theta; \mathbb{R})$ and this solution is strictly increasing as well as strictly convex.
  2. The fund loading function
     \[
     \phi^*(\theta) = \frac{1}{F(\theta, v^*(\theta), \dot{v}^*(\theta))} \left( \sqrt{2v^*(\theta) - [\dot{v}^*(\theta)]^2}, \dot{v}^*(\theta) \right)^T
     \]
     is well-defined for all investor types (i.e. $F > 0$ on $\Theta$) and generates an optimal fund menu.
  3. The function $\phi_1^*(\theta)$ is decreasing in type while the functions $\phi_2^*(\theta)$ and $\pi(\theta, \phi^*(\theta))$ are both increasing.
  4. The manager utility $M$ is increasing in both $\xi$ and $|\Theta|$. 
Investor welfare

Optimal fund menus
Optimality of bundling

- Since $\dot{v}^*(0) = 0$ the index is part of the optimal menu.

It is never optimal to engage in mixed bundling by also offering the non-index asset as part of the menu.

*Without bundling screening becomes impossible.*

- The optimality of offering funds rather than individual assets—i.e. bundling—is directly related to linear pricing.
1st best

• In the 1st best the manager offers

\[ \phi^\circ(\theta) = \frac{2\xi(\theta)}{\|\xi(\theta)\|^2} \Rightarrow (\pi^\circ(\theta), v^\circ(\theta)) = \left( \frac{\|\xi(\theta)\|^2}{4}, \frac{\|\xi(\theta)\|^2}{8} \right). \]

• Comparing to \( \phi^*(\theta) \), \( \pi(\theta, \phi^*(\theta)) \), and \( v^*(\theta) \) allows to elicit the impact of the information friction (under linear pricing).

• **Lemma.** (Closet indexing) The function

\[ \Delta(\theta) = \frac{\phi^\circ_2(\theta)}{\phi^\circ_1(\theta)} - \frac{\phi^*_2(\theta)}{\phi^*_1(\theta)} \]

is nonnegative, inverse \( u \)--shaped and equal to zero at both ends of the type space.
Risk exposure

Optimal fund menus
Utility loss

Utility loss $1 - \frac{M^p}{M}$ (%)

Risk premium $\xi$

Range $|\Theta|$
Utility and allocations

\[ v^*(\theta_H) \]

\[ v^0(\theta_H) \]

\[ v^0(0) \]

\[ v^*(0) \]

Type \( \theta \)

Base

1st Best

Optimal fund menus
Non–exclusivity

• Suppose that the manager can prevent investors from combining the funds in the menu.

• Incentive compatibility then requires that

\[ v(\theta) \geq v(\theta, \theta') = \max_{q \geq 0} u(\theta; r + q(\phi(\theta')^\top \epsilon - 1)) \]

for all pairs of investor types.

• The triangle inequality shows that, as expected, this condition is less strict than the non exclusive condition (IC).

• The 1st order conditions are the same. They lead to the same optimal menu, but that need not be true in general.
Direct index access

• If the fee rate $\gamma_I \geq \gamma^*_I$ then nothing changes.

• Otherwise, proceeding as in the benchmark case shows that the relaxed problem is now given by

$$\sup_v \int_{\Theta} F(\theta, \nu(\theta), \dot{\nu}(\theta)) \frac{d\theta}{|\Theta|}$$

subject to

$$\inf_{\theta \in \Theta} \left\{ 2\nu(\theta) - [\dot{\nu}(\theta)]^2 - (\xi - \gamma_I)^2 \right\} \geq 0.$$ 

• If the fee rate $\gamma_I \in (\xi/3, \gamma^*_I)$ the manager offers the index to all types below an endogenous cutoff $\theta^* \leq \frac{1}{3}|\Theta|$ but the solution is otherwise similar to that of the benchmark model.

• If the fee rate $\gamma_I \leq \xi/3$ then the optimal menu is **equivalent to offering access to the two assets** with fee rates $\gamma_I$ and $\frac{1}{3}|\Theta|$.
Future research?

- A screening problem with
  - multiple goods
  - flexible quantities
  - buyers can mix contracts
  - unobserved preferences on some of the goods
  - linear pricing
  - the seller can offer the products in bundles
Lemma. It is sufficient for investor $\theta$ to invest only in two funds. This is because the optimization problem of investor $\theta$ can be written as

$$v_i(\theta, m) = \sup_{x \in \mathbb{R}^2} \sup_{p \in \mu_+^x(M)} \left\{ x_1 \xi_1 + x_2 \theta - \frac{1}{2} a\|x\|^2 - \int_M \gamma(m)p(dm) \right\} \quad (10)$$

where

$$\mu_+^x(M) = \left\{ p \in \mu_+(M) : \int_M \phi(m)p(dm) = x \right\}. \quad (11)$$

Then, the result follows from standard deterministic optimization results.
Proof of Proposition 1.

Suppose investor of type $\theta \in \Theta$ allocates money to a pair of funds $(m_1(\theta), m_2(\theta))$. We need to choose the fund loading vector $\phi(\theta)$ so that

$$\sum_{k=1}^{2} \gamma_0(m_k(\theta)) p_k(\theta) = \pi(\theta, \phi(\theta)), \quad (12)$$

$$\frac{a}{2} \left\| \sum_{k=1}^{2} p_k(\theta) \phi_0(m_k(\theta)) \right\|^2 = \nu(\theta) = \frac{1}{2a\|\phi(\theta)\|^2} (\phi(\theta) \top \xi(\theta) - 1), \quad (13)$$

The solution is

$$\phi(\theta) = \frac{p_1(\theta) \phi_0(m_1(\theta)) + p_2(\theta) \phi_0(m_2(\theta))}{p_1(\theta) \gamma_0(m_1(\theta)) + p_2(\theta) \gamma_0(m_2(\theta))}. \quad (14)$$
Lemma. Assume that $v^* \in C^2(\Theta; \mathbb{R})$ is a solution to the boundary value problem. Then, $v^*$ is optimal for the relaxed problem.

Proof: Let $v$ be another feasible function. It can be shown that $F$ is concave, so that

$$
\int_{\Theta} \left( F(x, v(\theta), \dot{v}(\theta)) - F(x, v^*(\theta), \dot{v}^*(\theta)) \right) d\theta \leq \Delta(v, v^*)
$$

(15)

$$
\equiv \int_{\Theta} \left( (v(\theta) - v^*(\theta)) F_{v^*}(\theta)(\theta) + (\dot{v}(\theta) - \dot{v}^*(\theta)) F_{\dot{v}^*}(\theta)(\theta) \right) d\theta
$$

(16)

Integration by parts shows that

$$
\Delta(v, v^*) = \left( (v - v^*)(\theta) F_p^*(\theta) \right) \bigg|_{\theta=0}^{\theta_H} + \int_{\Theta} (v - v^*)(\theta) \left( F_v^*(\theta) - \frac{d}{d\theta} F_p^*(\theta) \right) d\theta
$$

(17)

$$
= (v(\theta_H) - v^*(\theta_H)) F_p^*(\theta_H) - (v(0) - v^*(0)) F_p^*(0) = 0
$$

(18)

where the last two equalities follow from the fact that $v^*$ solves the ODE and the boundary conditions.
Thank you for your attention!

For potential PhD students in Social Sciences at Caltech:
http://www.hss.caltech.edu/content/graduate-studies