Optimal fund menus

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Motivation and Preview

- Why do asset managers offer so many different mutual funds?
- What is the optimal menu of funds?
- This paper provides a novel model:

  The characteristic(s) of investors are unobserved.

  Fees are linear within a given class of mutual fund shares.

  ⇒ Bundling—i.e. offering funds rather than the underlying assets—is the optimal way to screen investors.

- This leads to a **new class of screening problems** that are subject to a linear pricing constraint.
- **Main results:**
  - an alternative explanation for "closet indexing" or "home bias".
  - linear pricing constraint increases investors' welfare

The model

- Static model with three assets:
  - A riskless asset with gross return r
  - Two risky assets whose gross excess returns ϵ are independent random variables with unit variance.
- One manager and unit mesure of risk-averse investors.
- Investors of type $\theta \in \Theta = [0, \theta_H]$ believe that $E_\theta[\epsilon] = \xi(\theta) = (\xi, \theta)^\top$ and their preferences are given by

$$u(\theta; w_1) = aE_\theta\left[w_1 - rw_0\right] - \frac{a^2}{2}\text{var}_\theta[w_1]$$

where $w_0 = 1$ and $a = 1$ are constants that represent the common initial wealth and risk aversion of investors.

The model

- Investors can only access the risky assets by investing in the funds offered by the manager (and paying the cost).

Q: What should the manager offer given that he only knows that types are uniformly distributed across investors?

- If any pricing scheme is allowed this is a standard screening problem for which the optimal strategy is to offer:
  1. A fixed cost for the index (no surplus for investors),
  2. A quantity dependent fee for the non index (excludes $\theta \leq \frac{1}{2}|\Theta|/2$).
- But nonlinear pricing is not allowed within a given class of shares.
- We solve for the manager's optimal offering strategy under a **linear pricing** constraint.

The model

- A (linearly priced) fund is specified by a pair $(\gamma, \phi) \in \mathbb{R}_+ \times \mathbb{R}^2$.
- A fund menu is a triple $\mathbf{m} = (\gamma, \phi, \mathcal{M})$ where $\mathcal{M}$ is an index set and $\gamma, \phi : \mathcal{M} \to \mathbb{R}_+ \times \mathbb{R}^2$ are functions that describe the funds.
- The best response of an investor of type θ to a fund menu $\mathbf{m}$ offered by the manager is the **measure**

$$q^*(\theta, \mathbf{m}) = \arg\max_{q \in \mu_+(\mathcal{M})} u\left(\theta; r + \int_{\mathcal{M}} \left(\phi(m)^\top \epsilon - \gamma(m)\right) q(dm)\right),$$

and the aggregation of these best responses generates

$$\int_{\mathcal{M} \times \Theta} \gamma(m) q^*(dm; \theta, \mathbf{m}) \frac{d\theta}{|\Theta|}$$

in fees for the manager.

The revelation principle

- Denote by

$$\pi(\theta, \phi) = \arg\max_{q \geq 0} u\left(\theta; r + q\left(\phi^{\top}\epsilon - 1\right)\right)$$

  the amount that an investor of type θ optimally allocates when the manager offers a single fund with characteristics $(1, \phi)$.

- **Proposition.** Given $\overline{\mathbf{m}}$ there exists $\mathbf{m}$ such that
  1. $\mathcal{M}(\mathbf{m}) = \Theta$
  2. $\gamma(\theta, \mathbf{m}) = 1$ for all $\theta \in \Theta$
  3. $q^*(dm; \theta, \mathbf{m}) = \pi(\theta, \phi(\theta))\delta_{\{\theta\}}(dm)$ for all $\theta \in \Theta$
  4. The manager and investors are indifferent between $\overline{\mathbf{m}}$ and $\mathbf{m}$

Incentive compatibility

- The problem reduces to maximizing

$$I(\phi) = \int_{\Theta} \pi\left(\theta, \phi(\theta)\right) \frac{d\theta}{|\Theta|}$$

over the set of fund loading functions that induce each investor to only allocate money to the fund that targets his type.

- **Proposition.** A fund loading function is **incentive compatible** if and only if it satisfies the inequality

$$\alpha(\theta'|\theta) := \phi(\theta')^\top \xi(\theta) - 1 - \frac{\phi(\theta')^\top \phi(\theta)}{\|\phi(\theta)\|^2}\left(\phi(\theta)^\top \xi(\theta) - 1\right)_+ \leq 0 \qquad \text{(IC)}$$

for all pairs of investor types $(\theta, \theta') \in \Theta$.

1st best benchmark

- Suppose that the manager knows the type of each investor.
- In this case his optimal strategy is to offer investors of type θ a single fund with unit fee and loadings

$$\phi^\circ(\theta) = \arg\max_{\phi \in \mathbb{R}^2} \pi(\theta, \phi) = \frac{2\xi(\theta)}{\|\xi(\theta)\|^2}.$$

- Substituting into (IC) shows that (for this menu):

$$\alpha(\theta'|\theta) = \frac{\theta'(\theta - \theta')}{\|\xi(\theta)\|^2} > 0, \qquad \theta' < \theta.$$

$\Rightarrow$ If the manager was to offer this menu each investor would have an **incentive to report a lower type than his own**.

Instruments

- The problem reduces to maximizing $I(\phi)$ over the set Φ of loading functions $\phi \in \mathrm{AC}(\Theta; \mathbb{R})$ that satisfy condition (IC).
- This problem is hard because the constraint is non standard and the optimization is over vector valued functions.
- **Lemma.** If $\phi \in \Phi$ then, with the value function

$$v(\theta) = \max_{q \geq 0} u\left(\theta; r + q\left(\phi(\theta)^\top \epsilon - 1\right)\right)$$

we have, from the first order condition, that,

$$\pi(\theta, \phi(\theta)) = \theta \dot{v}(\theta) - 2v(\theta) + \xi \sqrt{2v(\theta) - [\dot{v}(\theta)]^2}$$

for every investor type $\theta \in \Theta$.

The relaxed problem

- It follows that

$$\sup_{\phi \in \Phi} I(\phi) \leq \sup_{v} \int_{\Theta} \left(\theta \dot{v}(\theta) - 2v(\theta) + \xi \sqrt{2v(\theta) - [\dot{v}(\theta)]^2} \right) \frac{d\theta}{|\Theta|}$$

  where the **relaxed problem** on the right only takes into account the 1st order condition associated with (IC).

- The Euler-Lagrange equation is, with $F(\theta, v, \dot{v})$ denoting the integrand,

$$0 = F_{v(\theta)} - \frac{d}{d\theta} F_{\dot{v}(\theta)}, \qquad \text{on } \Theta, \qquad \text{(EL)}$$

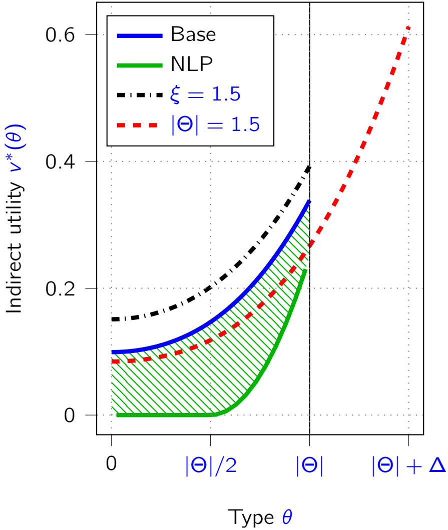
$$0 = F_{\dot{v}(\theta)}, \qquad \text{on } \partial\Theta. \qquad \text{(BC)}$$

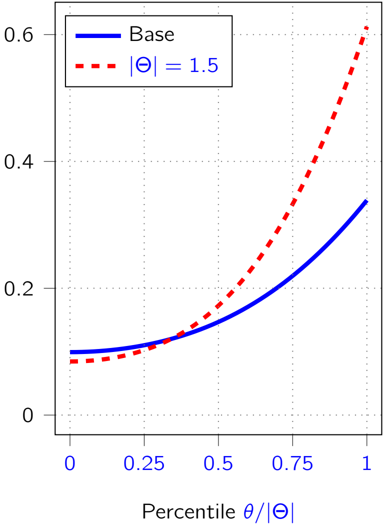
Because the integrand is concave in $(v, \dot{v})$ this Boundary Value Problem is necessary and sufficient for optimality in the relaxed problem.

The optimal fund menu

- **Main results.**
  1. (EL) and (BC) admit a unique solution $v^* \in C^2(\Theta; \mathbb{R})$ and this solution is strictly increasing as well as strictly convex.
  2. The fund loading function
  $$\phi^*(\theta) = \frac{1}{F(\theta, v^*(\theta), \dot{v}^*(\theta))} \left(\sqrt{2v^*(\theta) - [\dot{v}^*(\theta)]^2}, \dot{v}^*(\theta) \right)^\top \qquad (1)$$
  is well-defined for all investor types (i.e. $F > 0$ on Θ) and generates an optimal fund menu.
  3. The function $\phi_1^*(\theta)$ is decreasing in type while the functions $\phi_2^*(\theta)$ and $\pi(\theta, \phi^*(\theta))$ are both increasing.
  4. The manager utility M is increasing in both ξ and $|\Theta|$.

Investor welfare





Optimality of bundling

- Since $\dot{v}^*(0) = 0$ the index is part of the optimal menu.

It is never optimal to engage in mixed bundling by also offering the

non index asset as part of the menu

Without bundling screening becomes impossible.

- The optimality of offering funds rather than individual assets—i.e. bundling—is directly related to linear pricing.

1st best

- In the 1st best the manager offers

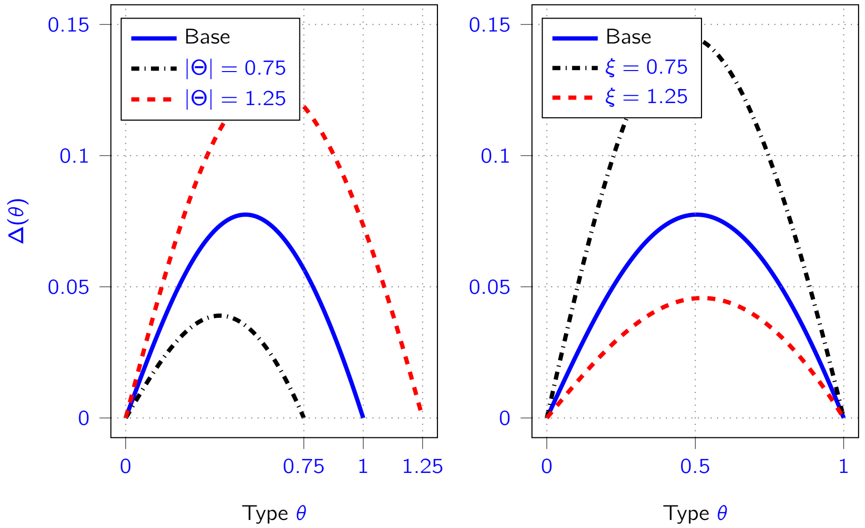
$$\phi^{\circ}(\theta) = \frac{2\xi(\theta)}{\|\xi(\theta)\|^2} \implies (\pi^{\circ}(\theta), v^{\circ}(\theta)) = \left(\frac{\|\xi(\theta)\|^2}{4}, \frac{\|\xi(\theta)\|^2}{8}\right).$$

- Comparing to $\phi^*(\theta)$, $\pi(\theta, \phi^*(\theta))$, and $v^*(\theta)$ allows to elicit the impact of the information friction (under linear pricing).
- **Lemma.** (Closet indexing) The function

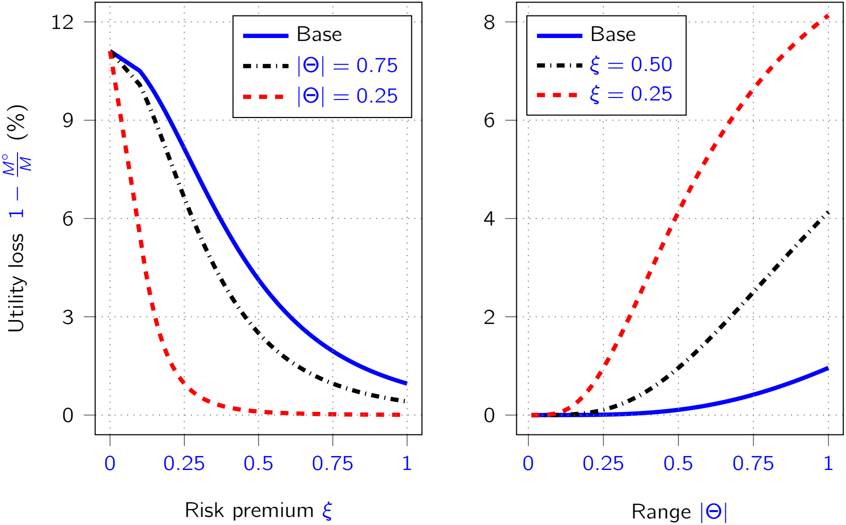
$$\Delta(\theta) = \frac{\phi_2^{\circ}(\theta)}{\phi_1^{\circ}(\theta)} - \frac{\phi_2^*(\theta)}{\phi_1^*(\theta)}$$

is nonnegative, inverse $u-$shaped and equal to zero at both ends of the type space.

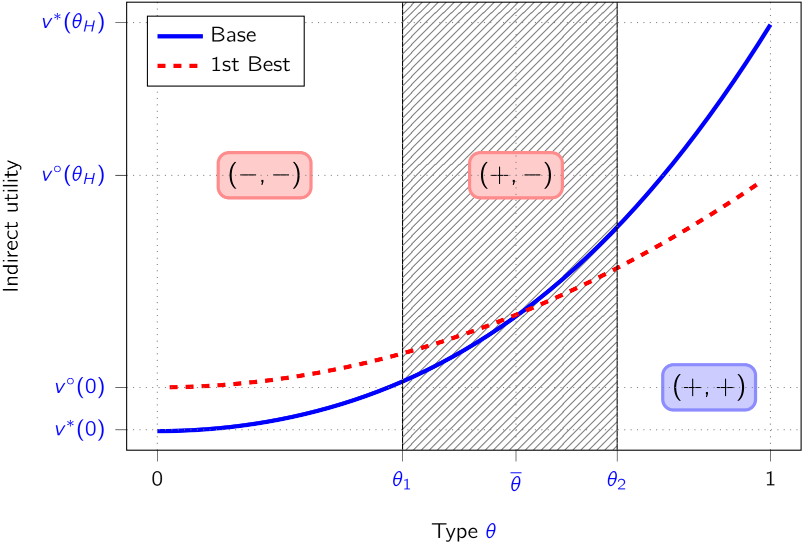
Risk exposure



Utility loss



Utility and allocations



Non–exclusivity

- Suppose that the manager can prevent investors from combining the funds in the menu.
- Incentive compatibility then requires that

$$v(\theta) \geq v(\theta, \theta') = \max_{q \geq 0} u\left(\theta; r + q\left(\phi(\theta')^\top \epsilon - 1\right)\right)$$

  for all pairs of investor types.
- The triangle inequality shows that, as expected, this condition is less strict than the non exclusive condition (IC).
- The 1st order conditions are the same. They lead to the same optimal menu, but that need not be true in general.

Direct index access

- If the fee rate $\gamma_I \geq \gamma_I^*$ then nothing changes.
- Otherwise, proceeding as in the benchmark case shows that the relaxed problem is now given by

$$\sup_v \int_\Theta F\left(\theta, v(\theta), \dot{v}(\theta)\right) \frac{d\theta}{|\Theta|}$$

subject to $\inf_{\theta \in \Theta} \left\{ 2v(\theta) - [\dot{v}(\theta)]^2 - (\xi - \gamma_I)^2 \right\} \geq 0.$

- If the fee rate $\gamma_I \in (\xi/3, \gamma_I^*)$ the managers offers the index to all types below an endogenous cutoff $\theta^* \leq \frac{1}{3}|\Theta|$ but the solution is otherwise similar to that of the benchmark model.
- If the fee rate $\gamma_I \leq \xi/3$ then the optimal menu is **equivalent to offering access to the two assets** with fee rates γ_I and $\frac{1}{3}|\Theta|$

Future research?

- A screening problem with
  - multiple goods
  - flexible quantities
  - buyers can mix contracts
  - unobserved preferences on some of the goods
  - linear pricing
  - the seller can offer the products in bundles

Proof of Proposition 1.

Lemma. It is sufficient for investor θ to invest only in two funds.
This is because the optimization problem of investor θ can be written as

$$v_i(\theta, \mathbf{m}) = \sup_{x \in \mathbb{R}^2} \sup_{p \in \mu_+^x(\mathbf{M})} \left\{ x_1 \xi_1 + x_2 \theta - \frac{1}{2} a \|x\|^2 - \int_{\mathbf{M}} \gamma(m) p(dm) \right\} \quad (10)$$

where

$$\mu_+^x(\mathbf{M}) = \left\{ p \in \mu_+(\mathbf{M}) : \int_{\mathbf{M}} \phi(m) p(dm) = x \right\}. \quad (11)$$

Then, the result follows from standard deterministic optimization results.

Proof of Proposition 1.

Suppose investor of type $\theta \in \Theta$ allocates money to a pair of funds $(m_1(\theta), m_2(\theta))$. We need to choose the fund loading vector $\phi(\theta)$ so that

$$\sum_{k=1}^{2} \gamma_0(m_k(\theta))p_k(\theta) = \pi(\theta, \phi(\theta)), \tag{12}$$

$$\frac{a}{2}\left\| \sum_{k=1}^{2} p_k(\theta)\phi_0(m_k(\theta)) \right\|^2 = v(\theta) = \frac{1}{2a\|\phi(\theta)\|^2}\left(\phi(\theta)^\top \xi(\theta) - 1\right), \tag{13}$$

The solution is

$$\phi(\theta) = \frac{p_1(\theta)\phi_0(m_1(\theta)) + p_2(\theta)\phi_0(m_2(\theta))}{p_1(\theta)\gamma_0(m_1(\theta)) + p_2(\theta)\gamma_0(m_2(\theta))}. \tag{14}$$

ODE

Lemma. Assume that $v^* \in C^2(\Theta; \mathbb{R})$ is a solution to the boundary value problem. Then, v^* is optimal for the relaxed problem.

Proof: Let v be another feasible function. It can be shown that F is concave, so that

$$\int_\Theta \Big(F\left(x, v(\theta), \dot{v}(\theta)\right) - F\left(x, v^*(\theta), \dot{v}^*(\theta)\right) \Big) d\theta \leq \Delta(v, v^*) \tag{15}$$

$$\equiv \int_\Theta \Big(\left(v(\theta) - v^*(\theta)\right) F^*_{v^*(\theta)}(\theta) + \left(\dot{v}(\theta) - \dot{v}^*(\theta)\right) F^*_{\dot{v}^*(\theta)}(\theta) \Big) d\theta \tag{16}$$

Integration by parts shows that

$$\Delta(v, v^*) = \left(\left(v - v^*\right)(\theta) F^*_p(\theta)\right)\Big|_{\theta=0}^{\theta_H} + \int_\Theta \left(v - v^*\right)(\theta) \left(F^*_v(\theta) - \frac{d}{d\theta} F^*_p(\theta) \right) d\theta \tag{17}$$

$$= \left(v(\theta_H) - v^*(\theta_H)\right) F^*_p(\theta_H) - \left(v(0) - v^*(0)\right) F^*_p(0) = 0 \tag{18}$$

where the last two equalities follow from the fact that v^* solves the ODE and the boundary conditions.

Thank you for your attention!

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