American options in a non-linear incomplete market model with default

Agnès Sulem

INRIA Paris
Equipe-Projet MATHRISK

Bachelier Finance Society One World seminar series
14 January 2021

joint work with M. GRIGOROVA (University of Leeds) & M.-C. QUENEZ (LPSM Paris)
Financial Market Features

- (Possible) **default** on the underlying risky asset.
- Market **imperfections**: the dynamics of the wealth process are **non-linear**.
  
  *Examples*: different lending and borrowing rates, different repo rates, impact of a large investor on the default intensity...

- The market is **incomplete**: *not every contingent claim is replicable.*
Goals

Study the seller’s (superhedging price) of an American option with irregular pay-off process (not necessarily right-continuous).
Introduction

Goals

Study the seller’s (superhedging price) of an American option with irregular pay-off process (not necessarily right-continuous).

Example (Non right-continuous Pay-off process)

- American digital put (resp. call) option (with strike $K > 0$).
  Corresponding payoff $\xi_t := 1_{S_t < K}$ (resp. $\xi_t := 1_{S_t \geq K}$).
- American call option with lower barrier, $\xi_t := (S_t - K)^+1_{\inf_{0 \leq s \leq t} S_s \geq L}$
Goals

Study the **seller’s (superhedging) price** of an American option with *irregular* pay-off process (**not** necessarily **right-continuous**).

- Dual representation (value of a non-linear mixed control/stopping problem)
- Characterization via a *constrained* reflected BSDE with default

Study the **buyer’s price**

- Dual representation (value of a non-linear control/stopping *game*)
Non-linear incomplete market with default
Probability setup

Let $(\Omega, G, P)$ be a complete probability space.

- Let $W$ be a one-dimensional Brownian motion.
- $\vartheta$ is a random variable which models the default time.
- Let $N$ be the default jump process defined by $N_t := 1_{\vartheta \leq t}$
- Let $G = \{ G_t, t \geq 0 \}$ be the augmented filtration generated by $W$ and $N$.

We assume that $W$ is a $G$-Brownian motion.
We have a $G$-martingale representation w.r.t. $W$ and $M$. (Jeanblanc-Song’15)

- Let $(\Lambda_t)$ be the $G$-predictable compensator of $(N_t)$.
  We assume that $\Lambda_t = \int_0^t \lambda_s ds$, $t \geq 0$, where $\lambda_s \geq 0$ is the intensity process. It vanishes after $\vartheta$. We suppose $\lambda$ bounded.
- Let $M$ be the compensated martingale of $(N_t)$ given by

\[ M_t := N_t - \Lambda_t. \]
Let $T > 0$ be the terminal time.

- $S^2 := \{ \text{adapted RCLL processes } X \text{ s.t. } \mathbb{E}[\sup_{0 \leq t \leq T} X_t^2] < +\infty \}$
- $H^2 := \{ \text{predictable processes } Z \text{ s.t. } \mathbb{E}\left[\int_0^T Z_t^2 dt\right] < \infty \}$
- $H^2_\lambda := \{ \text{predictable processes } K \text{ s.t. } \mathbb{E}\left[\int_0^T K_t^2 \lambda_t dt\right] < \infty \}$
The financial market $\mathcal{M}^f$

- one risk-free asset: $dS^0_t = S^0_tr_tdt$
- one risky asset with price process $S = (S_t)_{0 \leq t \leq T}$:

$$dS_t = S_t\left(\mu_t dt + \sigma_t dW_t + \beta_t dM_t\right)$$

with $S_0 > 0$.

The processes $\sigma_t$, $\mu_t$, and $\beta_t$ are $\mathbb{G}$-predictable bounded with $\sigma_t > 0$ and $\beta_0 > -1$.

To simplify the presentation, suppose $\sigma_t = 1$.
Investor endowed with an initial wealth $x$.

at each time $t$, he chooses the amount $Z_t$ invested in the risky asset (where $Z_t \in \mathbb{H}^2$),

Let $V_t^{x,Z}$ the value of the associated portfolio (wealth) at time $t$.

The wealth process $V_t^{x,Z}$ satisfies the following dynamics

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_0 = x.$$ 

where $f : (t, \omega, y, z) \mapsto f(t, \omega, y, z)$ is a (non-convex) Lipschitz driver and satisfies $f(t, 0, 0) = 0$. 
Examples

- **linear case**: \( f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t \), where \( \theta_t = \mu_t - r_t \).

- **different borrowing and lending interest rates** \( R_t \) and \( r_t \) (with \( R_t \geq r_t \)): \( f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t + (R_t - r_t)(V_t - Z_t)^- \).
  (cf. e.g. Crépey '15 in the case of CVA contracts with funding costs)

- **a repo market** on which the risky asset is traded
  \( f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t - l_t Z_t^- + b_t Z_t^+ \),
  \( b_t = \) borrowing repo rate; \( l_t = \) lending repo rate. (Brigo’16 et al, Bielecki-Rutkowski’15)

- **Effect of a large seller** on the default intensity
  (Dum.-Gri.-Quenez-S.’18), ...
The market $\mathcal{M}^f$ is **incomplete**:

2 sources of risk ($W$ and $M$); only 1 risky asset for investment.

Indeed, let $\eta \in L^2(G_T)$ be the terminal payoff of a European option. It might not be possible to find $(x, Z)$ in $\mathbb{R} \times \mathbb{H}^2$ such that

$$V^{x, Z}_T = \eta.$$

In other words, there does not necessarily exist $(V, Z) \in \mathbb{H}^2 \times \mathbb{H}^2$ such that

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_T = \eta.$$

∃! $(Y, Z, K)$ in $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ solution of the BSDE with default

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \eta \quad (2.1)$$

but in general $K \neq Z\beta$. 

(INRIA, Mathrisk)
American option superhedging

American option with maturity $T$ and payoff $(\xi_t) \in S^2$.

Here $S^2$ is the vector space of $\mathbb{R}$-valued optional (not necessarily RCLL) processes $\phi$ such that $\|\phi\|_{S^2}^2 := \mathbb{E}\left[\text{ess sup}_{\tau \in T} |\phi_\tau|^2\right] < \infty$. Note that $S^2$ is the sub-space of RCLL processes of $S^2$. 
Definition

A superhedge for the seller against the American option with initial price $x \in \mathbb{R}$ is a strategy $Z \in H^2$ s.t. $V_t^{x,Z} \geq \xi_t$, $0 \leq t \leq T$ a.s.

Let $\mathcal{A}(x) := \{\text{superhedges for the seller associated with } x\}$

Definition (Seller’s (superhedging) price at time 0)

$$u_0 := \inf \{ x \in \mathbb{R}, \exists Z \in \mathcal{A}(x) \}.$$
Dual representation of the seller’s price
The linear (incomplete) case with a RCLL payoff

Recall that in the case of the linear incomplete market, that is when
\[ f(t, y, z) = -r_t y - \theta_t z, \]
we have the following dual representation of \( u_0 \):

\[
 u_0 = \sup_{(R, \tau) \in \mathcal{P} \times \mathcal{T}} \mathbb{E}_R \left( e^{-\int_0^\tau r_s ds} \xi_\tau \right)
\]

where \( \mathcal{P} \) is the set of all martingale probability measures Kramkov ’96.

We recall that a probability \( R \sim P \) is a martingale probability measure if

- the discounted risky-asset price \( (e^{-\int_0^t r_s ds} S_t) \) is a martingale under \( R \)
- \( \Leftrightarrow \) for all \( x \in \mathbb{R}, Z \in \mathbb{H}^2 \), the discounted wealth process
  \( (e^{-\int_0^t r_s ds} V^x_t, Z) \) (where \( V^x_t, Z \) follows the dynamics with the linear driver) is a martingale under \( R \).
The nonlinear (incomplete) case with irregular payoff

Key tools for the dual representation

- Non-linear operator $\mathcal{E}_Q^f$
- A suitable set $\mathcal{Q}$ of equivalent probability measures (called $f$-martingale probability measures).
- $\mathcal{E}_Q^f$-optional decomposition for strong $\mathcal{E}_Q^f$-supermartingales, $\forall Q \in \mathcal{Q}$
Non-linear $f$-expectation under $Q$ : $\mathcal{E}_Q^f$

Let $Q$ be a probability measure equivalent to $P$. From the $\mathcal{G}$-martingale representation thm, its density process $(\zeta_t)$ satisfies

$$d\zeta_t = \zeta_t - (\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1,$$

where $(\alpha_t)$ and $(\nu_t)$ are $\mathcal{G}$-predictable processes with $\nu_{\varnothing \land T} > -1$ a.s.

By Girsanov's theorem, $W_t^Q := W_t - \int_0^t \alpha_s ds$ is a $Q$-Brownian motion, and $M_t^Q := M_t - \int_0^t \nu_s \lambda_s ds$ is a $Q$-martingale.

We have a representation for $Q$-martingales w.r.t. $W^Q$ and $M^Q$. 

(INRIA, Mathrisk)
**f**-expectation under $Q$ : $\mathcal{E}_Q^f$

We call $f$-expectation under $Q$, the operator $\mathcal{E}_Q^f$ defined by :
for $\xi \in L^2_Q(\mathcal{G}_T)$,
$$
\mathcal{E}_Q^{f,s,T}(\xi) := X_s, \quad s \in [0, T]
$$
where $(X, Z, K) \in S^2_Q \times H^2_Q \times H^2_{Q,\lambda}$ satisfies the $Q$-BSDE

$$
-dX_t = f(t, X_t, Z_t)dt - Z_t dW^Q_t - K_t dM^Q_t; \quad X_T = \xi.
$$
Definition

Let \( Y \in \mathcal{S}_Q^2 \). The process \((Y_t)\) is said to be a strong \( \mathcal{E}_Q^f \)-supermartingale, if \( \forall \sigma, \tau \in \mathcal{T} \) with \( \sigma \leq \tau \),

\[
\mathcal{E}_Q^f(Y_\tau) \leq Y_\sigma \quad \text{a.s.}
\]

In the case of equality, the process \((Y_t)\) is said to be a (strong) \( \mathcal{E}_Q^f \)-martingale.
Dual representation of the seller's price

The set $\mathcal{Q}$ of $f$-martingale probability measures

**Definition**

A probability $Q \sim P$ is called an $f$-martingale probability measure if:

$\forall x \in \mathbb{R}$ and $\forall Z \in \mathbb{H}_Q^2$, the wealth $V^{x,Z}$ is a $\mathcal{C}^f_Q$-martingale.

We denote by $\mathcal{Q} := \{ f$-martingale probability measures $\}$

- $\int (\sigma_t dW_t + \beta_t dM_t)$ is a $Q$-martingale $\forall Q \in \mathcal{Q}$
- $P \in \mathcal{Q}$

(INRIA, Mathrisk)
Theorem (Non-linear $\mathcal{E}^f$-optional decomposition G-Q-S-2018)

Let $(X_t) \in \bigcap_{Q \in \mathcal{Q}} S^2_Q$ (not necessarily RCLL). If $(X_t)$ is a strong $\mathcal{E}_Q^f$-supermartingale $\forall Q \in \mathcal{Q}$, then there exists $Z \in \mathbb{H}^2$, $C \in \mathbb{C}^2$, and a nondecreasing optional RCLL process $h$ with $h_0 = 0$ and $\mathbb{E}[h_T^2] < \infty$ s.t.

$$-dX_t = f(t, X_t, Z_t)dt - Z_t(dW_t + \beta_t dM_t) + dC_t - + dh_t.$$ 

This decomposition is unique.

Here $\mathbb{C}^2$ is the set of purely discontinuous non decreasing RCLL optional processes $C$ with $C_0^- = 0$ and $\mathbb{E}(C_T^2) < \infty$.

Remark : $C_t - C_{t-} = -(X_{t+} - X_t)$.
Seller’s price $v_0$ of a European option

Consider a European option with maturity $T$ and payoff $\xi \in \mathbb{L}^2(G_T)$. 
Superhedging price $v_0$:

$$v_0 := \inf \{ x \in \mathbb{R} : \exists \phi \in H^2 \text{ s.t. } V_T^{x;\phi} \geq \xi \text{ a.s.} \}.$$

Hyp : $\exists x \in \mathbb{R}$ and $\psi \in H^2$ s.t. $\xi \leq V_T^{x,\psi}$ a.s. ($\iff v_0 < +\infty$).

**Theorem (Pricing-hedging duality)**

Suppose $\xi \in \bigcap_{Q \in \mathcal{Q}} \mathbb{L}^2_Q$. The superhedging price $v_0$ of the European option satisfies

$$v_0 = \sup_{Q \in \mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\xi).$$
Seller’s price $u_0$ of the American option

Consider an American option with maturity $T$ and payoff $(\xi_t) \in S^2$. Assume that there exist $x \in \mathbb{R}$ and $\psi \in H^2$ satisfying for all $0 \leq t \leq T$

$$\xi_t \leq V_t^x,\psi$$
Seller’s price $u_0$ of the American option

Consider an American option with maturity $T$ and payoff $(\xi_t) \in S^2$. Assume that there exist $x \in \mathbb{R}$ and $\psi \in H^2$ satisfying for all $0 \leq t \leq T$

$$\xi_t \leq V_t^{x,\psi}$$

**Theorem (Pricing-hedging duality)**

Suppose $(\xi_t) \in \cap_{Q \in \mathcal{Q}} S^2_Q$. The superhedging price $u_0$ of the American option satisfies

$$u_0 = \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^f_{Q,0,\tau}(\xi_\tau).$$
Sketch of the proof
Recall that \( u_0 := \inf \{ x \in \mathbb{R}, \exists Z \in \mathbb{H}^2 \text{ s.t. } V^{x,Z} \geq \xi. \} \).

- Proof of the inequality:

\[
   u_0 \geq \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{Q,0,\tau}(\xi_{\tau}). \tag{3.1}
\]

Let \( x \in \mathbb{R} \) be such that \( \exists Z \in \mathbb{H}^2 \) with \( V^{x,Z} \geq \xi \).

By definition, for each \( f \)-martingale probability measure \( Q \), the wealth process \( V^{x,Z} \) is an \( \mathcal{E}_Q \)-martingale. The result easily follows. (we have \( x = V_0^{x,Z} = \mathcal{E}_{Q,0,\tau}(V^{x,Z}_\tau) \geq \mathcal{E}_{Q,0,\tau}(\xi_{\tau}). Hence, \( x \geq \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{Q,0,\tau}(\xi_{\tau}), which implies (3.1)\))
Sketch of the proof
Recall that \( u_0 := \inf \{ x \in \mathbb{R}, \exists Z \in \mathbb{H}^2 \text{ s.t. } V^{x,Z} \geq \xi \} \).

- Proof of the inequality:

\[
u_0 \geq \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}^E_Q} \mathcal{E}_{Q,0,\tau}(\xi_{\tau}). \tag{3.1}
\]

Let \( x \in \mathbb{R} \) be such that \( \exists Z \in \mathbb{H}^2 \) with \( V^{x,Z} \geq \xi \).

By definition, for each \( f \)-martingale probability measure \( Q \), the wealth process \( V^{x,Z} \) is an \( \mathcal{E}_{Q} \)-martingale. The result easily follows. (we have \( x = V^x_0, Z = \mathcal{E}_{Q,0,\tau}(V^{x,Z}_{\tau}) \geq \mathcal{E}_{Q,0,\tau}(\xi_{\tau}). \) Hence, \( x \geq \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}^E_Q} \mathcal{E}_{Q,0,\tau}(\xi_{\tau}), which implies (3.1)\)

- Proof of the converse inequality
To this aim, we introduce the dual value function at time $S \in \mathcal{T}$

$$Y(S) := \text{ess sup}_{Q \in \mathcal{Q}} \text{ess sup}_{\tau \geq S} \mathcal{E}_{Q,S,\tau}(\xi_{\tau})$$  \hspace{1cm} (3.2)$$

The study of this optimization problem is quite technical. Let us first recall the simpler case when the market is complete.
The case of a complete market

Suppose $\beta = 0$ and the (Brownian) filtration is associated with $W$. The market is complete and $\mathcal{Q} = \{ P \}$. In this case, the dual value is given by the following optimal stopping problem with $\mathcal{E}^f$-expectation at time $S \in \mathcal{T}$

$$Y(S) := \operatorname{esssup}_{\tau \geq S} \mathcal{E}_{P,S,\tau}^f(\xi_\tau) \quad (3.3)$$

For each $S \in \mathcal{T}$, $Y(S) = Y_S$ a.s., where $Y$ is equal to the solution of the reflected BSDE with obstacle $\xi$ and driver $f$ (in the sense of Grig-I-O-Quen. (18)), that is $Y \in \mathcal{S}^2$ with $Y_t \geq \xi_t$ and $\exists! (Z, K) \in \mathcal{H}^2 \times \mathcal{H}^2_\lambda$, a predictable process $A \in \mathcal{A}^2$ s.t.

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + dC_{t^-}; \quad Y_T = \xi_T; \quad (Y_t - \xi_t)(A_t - A_{t^-}) = 0 \quad \text{(Skorokhod cond.)};$$

$$\quad (Y_t - \xi_t)(C_t - C_{t^-}) = 0 \quad \text{(Skorokhod cond.)}$$
The case of a complete market

Using this characterisation of the value of the above non-linear optimal stopping problem, we can show that the superhedging price $u_0$ of the American option satisfies

$$u_0 = Y_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^f_{P,0,\tau}(\xi_\tau).$$

Note that here, $P$ is the (unique) $f$-martingale probability measure.
The value function in the incomplete case

In this case, the dual value function at time $S \in \mathcal{T}$ is given by

$$Y(S) := \text{ess sup}_{Q \in \mathcal{Q}} \, \text{ess sup}_{\tau \geq S} \, E_{Q,S,\tau}(\xi_{\tau})$$
The value function in the incomplete case

In this case, the dual value function at time $S \in \mathcal{T}$ is given by

$$Y(S) := \text{ess sup}_{Q \in \mathcal{Q}} \text{ess sup}_{\tau \geq S} \mathcal{E}_{Q,S,\tau}(\xi\tau) = \text{ess sup}_{Q \in \mathcal{Q}} Y^Q_S \quad \text{a.s.} \quad (3.4)$$

where for each $Q \in \mathcal{Q}$,

$$Y^Q_S = \text{ess sup}_{\tau \geq S} \mathcal{E}_{Q,S,\tau}(\xi\tau);$$

In other terms, $Y^Q_S$ is the value function of the optimal stopping problem with $\mathcal{E}_{Q}$-expectation and with payoff $\xi$.

We know that the process $Y^Q$ is equal to the solution of the reflected BSDE under $Q$ associated with obstacle $\xi$ and driver $f$. Hence, by (3.4), the dual value function $Y(S)$ is equal to the ess. sup. of a family of reflected BSDEs.

(INRIA, Mathrisk)
We have $\mathbb{E}[\text{ess sup}_{S \in \mathcal{T}} Y(S)^2] < \infty$.

There exists a right-u.s.c process $(Y_t) \in \mathcal{S}^2$, called the value process s.t. $\forall S \in \mathcal{T}, Y_S = Y(S)$ a.s.
We have $\mathbb{E}[\text{ess sup}_{S \in \mathcal{T}} Y(S)^2] < \infty$.

There exists a right-u.s.c process $(Y_t) \in \mathcal{S}^2$, called the value process s.t. $\forall S \in \mathcal{T}, \ Y_S = Y(S)$ a.s.

$Y$ is a strong $\mathcal{E}_Q$-supermartingale for all $Q \in \mathcal{Q}$, with $Y \geq \xi$ and $Y_T = \xi_T$. It is also the minimal one.

---

(INRIA, Mathrisk)
We have $\mathbb{E}[^{\operatorname{ess sup}}_{S \in \mathcal{T}} Y(S)^2] < \infty$.

There exists a right-u.s.c process $(Y_t) \in \mathcal{S}^2$, called the value process s.t. $\forall S \in \mathcal{T}$, $Y_S = Y(S)$ a.s.

$Y$ is a strong $\mathcal{E}_Q$-supermartingale for all $Q \in \mathcal{Q}$, with $Y \geq \xi$ and $Y_T = \xi_T$. It is also the minimal one.

Hence, $Y$ admits an $\mathcal{E}^f$-optional decomposition : 

$\exists Z \in \mathbb{H}^2$, $\exists C \in \mathbb{C}^2$, and a nondecreasing optional process $h$ s.t.

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t - C_{t-}, \ 0 \leq t \leq T.$$
We have $\mathbb{E}[\text{ess sup}_{S \in \mathcal{T}} Y(S)^2] < \infty$.

There exists a right-u.s.c process $(Y_t) \in \mathcal{S}^2$, called the value process s.t. $\forall S \in \mathcal{T}, Y_S = Y(S)$ a.s.

$Y$ is a strong $\mathcal{E}_Q$-supermartingale for all $Q \in \mathcal{Q}$, with $Y \geq \xi$ and $Y_T = \xi_T$. It is also the minimal one.

Hence, $Y$ admits an $\mathcal{E}^f$-optional decomposition : $\exists Z \in \mathbb{H}^2, \exists C \in \mathbb{C}^2$, and a nondecreasing optional process $h$ s.t.

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t - C_t^-, 0 \leq t \leq T.$$ 

By a **forward** comparison theorem, $V.^{Y_0, Z} \geq Y. (\geq \xi.)$

Hence $Z$ is a **superhedging strategy** (associated with $Y_0$), which implies $Y_0 \geq u_0$ and hence $Y_0 = u_0 = \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{Q,0,\tau}(\xi_{\tau})$. Q.E.D
We have $\mathbb{E}[\text{ess sup}_{S \in \mathcal{T}} Y(S)^2] < \infty$.

There exists a right-u.s.c process $(Y_t) \in \mathcal{S}^2$, called the value process s.t. $\forall S \in \mathcal{T}, Y_S = Y(S)$ a.s.

$Y$ is a strong $\mathcal{E}_Q$-supermartingale for all $Q \in \mathcal{Q}$, with $Y \geq \xi$ and $Y_T = \xi_T$. It is also the minimal one.

Hence, $Y$ admits an $\mathcal{E}^f$-optional decomposition:

$\exists Z \in \mathbb{H}^2, \exists C \in \mathbb{C}^2$, and a nondecreasing optional process $h$ s.t.

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) ds + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t - C_{t-}, 0 \leq t \leq T.$$  

By a forward comparison theorem, $V^{Y_0, Z} \geq Y. (\geq \xi.)$

Hence $Z$ is a superhedging strategy (associated with $Y_0$), which implies $Y_0 \geq u_0$ and hence

$Y_0 = u_0 = \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{Q, 0, \tau}(\xi_\tau)$. Q.E.D

The process $(h_t + C_{t-})$ can be interpreted as the cumulative amount the seller withdraws from the hedging portfolio up to time $t$. 

(INRIA, Mathrisk)
Characterization of the seller’s price via a constrained reflected BSDE
We first provide a \textbf{predictable} $\mathcal{E}^{f}$-decomposition for strong $\mathcal{E}^{f}_{Q}$-supermartingale $\forall \, Q \in \mathcal{Q}$.

\textbf{Theorem}

Let $(X_t) \in \bigcap_{Q \in \mathcal{Q}} \mathbb{S}^2_Q$ be a strong $\mathcal{E}^{f}_{Q}$-supermartingale $\forall \, Q \in \mathcal{Q}$. There exists a unique process $(Z, K, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathcal{C}^2$ such that

$$- dX_t = f(t, X_t, Z_t) \, dt - Z_t \, dW_t - K_t \, dM_t + dA_t + dC_t -$$

$$A_0 + \int_0^t (K_s - \beta_s Z_s) \lambda_s \, ds \in \mathcal{A}^2$$

$$(K_t - \beta_t Z_t) \lambda_t \leq 0, \, t \in [0, T], \, dP \otimes dt - \text{a.e.}$$

where $\mathcal{A}^2 = \{\text{predictable RCLL non-decreasing processes } A \text{ with } A_0 = 0 \text{ and } \mathbb{E}(A_T^2) < +\infty\}$

\textbf{Remark} : $h_t = A_t - \int_0^t (K_s - \beta_s Z_s) \, dM_s$. 
European case

Theorem (European seller’s price)

\[ v_0 = X_0, \]

where \((X_t)\) is a \((RCLL)\) supersolution of a \textbf{constrained BSDE}, that is \(X \in S^2\) and \(\exists (Z, K) \in H^2 \times H^2_\lambda\) and a \textbf{predictable} nondecreasing RCLL process \(A\) with \(A_0 = 0\) and \(\mathbb{E}(A_T^2) < \infty\) such that

\[-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t; \quad X_T = \xi; \]

\[ A \cdot \int_0^t (K_s - \beta_s Z_s) \lambda_s ds \quad \text{is nondecreasing and} \]

\[(K_t - \beta_t Z_t) \lambda_t \leq 0, \quad dP \otimes dt - \text{a.e.}; \]

and it is the \textit{minimal} process which satisfies these properties.
Theorem (American seller's price)

\[ u_0 = Y_0, \]

where \( (Y_t) \) is a supersolution of the constrained \textit{reflected} BSDE that is, \( Y \in S^2 \) with \( Y_t \geq \xi_t \) and

\[ \exists! (Z, K) \in H^2 \times H^2_{\lambda}, \text{a predictable process } A \in A^2 \text{ and a purely discontinuous nondecreasing RCLL optional process } C \in C^2 \text{ s.t.} \]

\[ -dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + dC_t^-; \quad Y_T = \xi_T; \]

\[ A. + \int_0^T (K_s - \beta_s Z_s)\lambda_s ds \text{ is nondecreasing} \]

\[ K_0 - \beta_0 Z_0 \leq 0 \text{ a.s.} \]

\[ (Y_t - \xi_t)(C_t - C_{t-}) = 0 \quad \text{(Skorokhod cond.)} \]

and it is the \textit{minimal} process which satisfies these properties.
Theorem (American seller’s price)

\[ u_0 = Y_0, \]

where \((Y_t)\) is a supersolution of the constrained reflected BSDE that is, \(Y \in S^2\) with \(Y_t \geq \xi_t\) and

\[ \exists! (Z, K) \in H^2 \times H^2_\lambda, \text{ a predictable process } A \in \mathcal{A}^2 \text{ and a purely discontinuous nondecreasing RCLL optional process } C \in \mathcal{C}^2 \text{ s.t.} \]

\[-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + dC_t^-; \quad Y_T = \xi_T; \]

\[ A. + \int_0^T (K_s - \beta_s Z_s)\lambda_s ds \text{ is nondecreasing} \]

\[ K_0 - \beta_0 Z_0 \leq 0 \text{ a.s.} \]

\[ (Y_t - \xi_t)(C_t - C_{t^-}) = 0 \quad \text{(Skorokhod cond.)} \]

and it is the minimal process which satisfies these properties.

Comparing to the complete case, there are two additional constraints, and the process \(A\) does not necessarily satisfies the Skorokhod cond.
Proof

We apply the $\mathcal{E}^f$-predictable decomposition theorem to the value process $(Y_t)$. (Recall $Y_S = \text{ess sup}_{Q \in \mathcal{Q}, \tau \geq S} \mathcal{E} Q, S, \tau(\xi_\tau)$.)
Proof

- We apply the $\mathcal{E}^f$-predictable decomposition theorem to the value process $(Y_t)$. (Recall $Y_S = \text{ess} \sup_{Q \in \mathcal{Q}, \tau \geq S} \mathcal{E}_{Q,S,\tau}(\xi_\tau)$.

- It remains to show the Skorokhod cond. : $(Y_S - \xi_S)(C_S - C_{S-}) = 0$ for all $S \in \mathcal{T}$. (i.e. the process $C$ increases only when $Y = \xi$).
Proof

- We apply the $\mathcal{F}^f$-predictable decomposition theorem to the value process $(Y_t)$. (Recall $Y_S = \text{ess sup}_{Q \in \mathcal{Q}, \tau \geq S} \mathcal{E}^Q, S, \tau(\xi_\tau)$.
- It remains to show the Skorokhod cond. : $(Y_S - \xi_S)(C_S - C_{S-}) = 0$ for all $S \in \mathcal{T}$. (i.e. the process $C$ increases only when $Y = \xi$).

Recall that $C_S - C_{S-} = -(Y_{S+} - Y_S)$.

Hence, it is sufficient to show that $(Y_S - \xi_S)(Y_{S+} - Y_S) = 0$. To this aim, we introduce the "strict value" function at time $S \in \mathcal{T}$,

$$Y^+(S) = \text{ess sup}_{Q \in \mathcal{Q}, \tau > S} \mathcal{E}^Q, S, \tau(\xi_\tau).$$
Proof

- We apply the $\mathcal{G}^f$-predictable decomposition theorem to the value process $(Y_t)$. (Recall $Y_S = \text{ess sup}_{Q \in \mathcal{Q}, \tau \geq S} \mathcal{G}_{Q,S,\tau}(\xi_\tau)$.)

- It remains to show the Skorokhod cond.: $(Y_S - \xi_S)(C_S - C_{S-}) = 0$ for all $S \in \mathcal{T}$. (i.e. the process $C$ increases only when $Y = \xi$).

Recall that $C_S - C_{S-} = -(Y_{S+} - Y_S)$.

Hence, it is sufficient to show that $(Y_S - \xi_S)(Y_{S+} - Y_S) = 0$. To this aim, we introduce the "strict value" function at time $S \in \mathcal{T}$,

$$Y^+(S) = \text{ess sup}_{Q \in \mathcal{Q}, \tau > S} \mathcal{G}_{Q,S,\tau}(\xi_\tau).$$

Lemma: For all $S \in \mathcal{T}$, $Y^+(S) = Y_{S+}$ a.s. (i.e. the family $(Y^+(S)$ is aggregated by the process of right limits of the value process $Y$) and

$$Y_S = Y_{S+} \vee \xi_S \text{ a.s.}$$

$$\rightarrow (Y_S - \xi_S)(Y_{S+} - Y_S) = 0 \text{ a.s. for all } S \in \mathcal{T}. \text{ Q.E.D.}$$
Buyer’s price of the American option

Maximal price which allows the buyer to construct a portfolio strategy and an exercise time so that he is superhedged.

**Definition**

A superhedge for the buyer against the American option with initial price \( x \in \mathbb{R} \) is a pair \((\tau, Z) \in \mathcal{T} \times \mathbb{H}^2 \) s.t. \( V_{\tau}^{-x, Z} + \xi_\tau \geq 0 \) a.s.

Starting with amount \(-x\) at \( t = 0\) and following the strategy \( Z\), the payoff received at \( \tau \) allows the buyer to recover the debt incurred at \( t = 0\) by buying the option.

Let \( \mathcal{B}(x) := \{\text{superhedges for the buyer associated with } x\} \)

**Definition (buyer’s (superhedging) price)**

\[ \tilde{u}_0 = \sup\{ x \in \mathbb{R}, \, \exists (\tau, Z) \in \mathcal{B}(x) \}. \]
Theorem

Suppose that there exist $x' \in \mathbb{R}$ and $\psi' \in H^2$ satisfying $-\xi_t \leq V_t^{-x',\psi'}$, and that $(\xi_t)$ is RCLL and left-u.s.c. along stopping times.

The buyer’s price $\tilde{u}_0$ satisfies

$$\tilde{u}_0 = \inf_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \{ -\mathcal{E}^f_Q,0,\tau(-\xi_\tau) \} = \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{Q}} \{ -\mathcal{E}^f_Q,0,\tau(-\xi_\tau) \}.$$ 

Moreover, there exists a superhedge $(\tilde{\tau}, \tilde{Z})$ for the buyer.
Theorem

Suppose that there exist $x' \in \mathbb{R}$ and $\psi' \in \mathbb{H}_2$ satisfying $-\xi_t \leq V_t^{\psi',x'}$, and that $(\xi_t)$ is RCLL and left-u.s.c. along stopping times.

The buyer's price $\tilde{u}_0$ satisfies

$$\tilde{u}_0 = \inf_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \{-E^f_{Q,0,\tau}(-\xi_\tau)\} = \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{Q}} \{-E^f_{Q,0,\tau}(-\xi_\tau)\}.$$

Moreover, there exists a superhedge $(\tilde{\tau}, \tilde{Z})$ for the buyer.

Idea of the proof: study of the dual (game) problem: for each $S \in \mathcal{T}$,

$$\tilde{Y}(S) = \text{ess inf}_{Q \in \mathcal{Q}} \text{ess sup}_{\tau \geq S} -E^f_{Q,S,\tau}(-\xi_\tau).$$

We introduce $\tilde{\tau} := \inf\{t \in [0, T] : \tilde{Y}_t = \xi_t\}$ and prove there exists a portfolio strategy $\tilde{\phi} \in \mathbb{H}_2$ such that $(\tilde{\tau}, \tilde{\phi})$ is a superhedge for the buyer.

Remark: the result still holds when, instead of being RCLL on $[0, T]$, $\xi$ is only right-u.s.c., and RCLL only at $\tilde{\tau}$. 

(INRIA, Mathrisk)
In the proof, in order to study the properties of the dual value $\tilde{Y}$ (in particular a dynamic programming principle), we note that

$$\tilde{Y}(S) = \operatorname{ess} \inf_{Q \in \mathcal{Q}} \operatorname{ess} \sup_{\tau \geq S} -\mathcal{E}_{Q,S,\tau}(-\xi_\tau) = \operatorname{ess} \inf_{Q \in \mathcal{Q}} \tilde{Y}_S^Q \quad \text{a.s.} \quad (5.1)$$

where, for each $Q \in \mathcal{Q}$,

$$\tilde{Y}_S^Q = \operatorname{ess} \sup_{\tau \in \mathcal{T}} -\mathcal{E}_{Q,S,\tau}(-\xi_\tau)$$

Now, $\tilde{Y}_S^Q$ can be shown to be equal to the solution of a reflected BSDE under $Q$ with lower obstacle $\xi$. (associated with an appropriate driver $\tilde{f}$). Hence, by (5.1), the dual value $\tilde{Y}(S)$ is equal to the $\operatorname{ess} \inf$ of a family of reflected BSDEs.

Contrary to the seller's case (for which we had an $\operatorname{ess. sup}$), we have here an $\operatorname{ess. inf}$, which leads to some specific technical problems.
Buyer’s nearly superhedging price

When $\xi$ does not satisfy any regularity assumption on the left, the value of this game can be interpreted as a nearly superhedging price, defined as the supremum of the initial prices which, for each $\varepsilon > 0$, allow the buyer to be $\varepsilon$–superhedged, that is

$$\tilde{u}_0 = \sup \left\{ z \in \mathbb{R}, \forall \varepsilon > 0, \exists (\tau, \varphi) \in \mathcal{T} \times \mathbb{H}^2 \text{ s.t. } V_{\tau}^{-z,\varphi} \geq -\xi_\tau - \varepsilon \text{ a.s.} \right\}.$$ 

$$\bar{u}_0 \geq \tilde{u}_0 \text{ (the buyer’s superhedging price) (we may have } \tilde{u}_0 = -\infty).$$
No arbitrage considerations

**Definition**

Let \( x \in \mathbb{R} \). Let \( y \in \mathbb{R} \) and \( \varphi \) in \( \mathbb{H}^2 \). We say that \((y, \varphi)\) is an arbitrage opportunity for the seller of the American option with initial price \( x \) if

\[
y < x \quad \text{and} \quad V_t^{y,\varphi} - \xi_t \geq 0 \quad \text{a.s. for all } \tau \in T.
\]

**Definition**

Let \( x \in \mathbb{R} \). Let \( y \in \mathbb{R} \), \( \tau \in T \) and \( \varphi \in \mathbb{H}^2 \). We say that \((y, \tau, \varphi)\) is an arbitrage opportunity for the buyer of the American option with initial price \( x \) if

\[
y > x \quad \text{and} \quad V_t^{-y,\varphi} + \xi_t \geq 0 \quad \text{a.s.}
\]
Proposition: Let $x \in \mathbb{R}$. There exists an arbitrage opportunity for the seller (resp. for the buyer) of the American option with price $x$ if and only if $x > u_0$ (resp. $x < \tilde{u}_0$).

Definition
A real number $x$ is called an *arbitrage-free price for the American option* if there exists no arbitrage opportunity, neither for the seller nor for the buyer.

Proposition: If $u_0 < \tilde{u}_0$, there does not exist any arbitrage-free price for the American option. If $u_0 \geq \tilde{u}_0$, the interval $[\tilde{u}_0, u_0]$ is the set of all arbitrage-free prices. We call it the *arbitrage-free* interval for the American option.
Conclusion

Key results

In a non-linear incomplete market with default, we have given

- a **dual representation of the seller’s superhedging price** of the American option (with completely irregular payoff) in terms of the value of a non-linear mixed control/stopping problem. The dual representation involves a suitable set of equivalent probability measures $Q \in \mathcal{Q}$, which we call $f$-martingale probability measures.

- Characterization of the seller’s superhedging price as the **minimal supersolution of a constrained reflected BSDE** with default.

- a **duality representation for the buyer’s price** in terms of the value of a non-linear control/stopping **game** problem.
The talk is based on these 2 papers:


Grigorova M., Quenez M.-C. and A. Sulem : American options in a non-linear incomplete market model with default, https://hal.archives-ouvertes.fr/hal-02025835/document

The complete case with defaultable asset with total default is treated in:


General study of BSDEs with default jump can be found in: