American options in a non-linear incomplete market model with default

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Financial Market Features

- (Possible) default on the underlying risky asset.
- Market imperfections : the dynamics of the wealth process are non-linear.

  Examples : different lending and borrowing rates, different repo rates, impact of a large investor on the default intensity...
- The market is incomplete : *not every contingent claim is replicable*.

Goals

Study the seller's (superhedging price) of an American option with *irregular* pay-off process (**not** necessarily **right-continuous**).

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Example (Non right-continuous Pay-off process)

- *American digital put (resp. call) option* (with strike $K > 0$).
  Corresponding payoff $\xi_t := \mathbf{1}_{S_t < K}$ (resp. $\xi_t := \mathbf{1}_{S_t \geq K}$).
- *American call option with lower barrier*, $\xi_t := (S_t - K)^+ \mathbf{1}_{\inf_{0 \leq s \leq t} S_s \geq L}$

Goals

Study the seller's (superhedging) price of an American option with *irregular* pay-off process (**not** necessarily **right-continuous**).

- Dual representation (value of a non-linear mixed control/stopping problem)
- Characterization via a *constrained* reflected BSDE with default

Study the buyer's price

- Dual representation (value of a non-linear control/stopping *game*)

Non-linear incomplete market with default

Probability setup

Let $(\Omega, \mathcal{G}, P)$ be a complete probability space.

- Let W be a one-dimensional Brownian motion.
- ϑ is a random variable which models the default time.
- Let N be the default jump process defined by $N_t := \mathbf{1}_{\vartheta \leq t}$
- Let $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ be the augmented filtration generated by W and N.

We assume that W is a $\mathbb{G}$-Brownian motion.

We have a $\mathbb{G}$-martingale representation w.r.t. W and M. (Jeanblanc-Song'15)

- Let (Λ_t) be the $\mathbb{G}$-predictable compensator of (N_t).
  We assume that $\Lambda_t = \int_0^t \lambda_s ds$, $t \geq 0$, where $\lambda_s \geq 0$ is the intensity process. It vanishes after ϑ. We suppose λ bounded.
- Let M be the compensated martingale of (N_t) given by

$$M_t := N_t - \Lambda_t \,.$$

Let $T > 0$ be the terminal time.

- $S^2 := \{$ adapted RCLL processes X s.t. $\mathbb{E}[\sup_{0\le t\le T} X_t^2] < +\infty\}$
- $\mathbb{H}^2 := \{$ predictable processes Z s.t. $\mathbb{E}\left[\int_0^T Z_t^2 dt\right] < \infty\}$
- $\mathbb{H}^2_\lambda := \{$ predictable processes K s.t. $\mathbb{E}\left[\int_0^T K_t^2 \lambda_t dt\right] < \infty\}$

The financial market $\mathcal{M}^f$

- one risk-free asset : $dS_t^0 = S_t^0 r_t dt$
- one risky asset with price process $S = (S_t)_{0 \le t \le T}$:

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dM_t) \text{ with } S_0 > 0.$$

The processes σ_t, μ_t, and β_t are $\mathbb{G}$- predictable bounded with $\sigma_t > 0$ and $\beta_\vartheta > -1$.

To simplify the presentation, suppose $\sigma_t = \mathbf{1}$

- Investor endowed with an initial wealth x.
- at each time t, he chooses the amount $\mathbf{Z_t}$ invested in the risky asset (where $\mathbf{Z_t} \in \mathbb{H}^2$),
- Let $V_t^{x,Z}$ the value of the associated portfolio (wealth) at time t.
- The wealth process $V_t^{x,Z}$ satisfies the following dynamics

$$-dV_t = \mathbf{f}(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_0 = x.$$

where $\mathbf{f} : (t, \omega, y, z) \mapsto \mathbf{f}(t, \omega, y, z)$ is a (non-convex) Lipschitz driver and satisfies $f(t, 0, 0) = 0$.

Examples

- **linear case** : $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t$, where $\theta_t = \mu_t - r_t$.
- **different borrowing and lending interest rates** R_t and r_t (with $R_t \geq r_t$) :
  $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t + (\mathbf{R_t} - \mathbf{r_t})(\mathbf{V_t} - \mathbf{Z_t})^-$.
  (cf. e.g. Crépey '15 in the case of CVA contracts with funding costs)
- **a repo market** on which the risky asset is traded

  $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t - \mathbf{l_t} \mathbf{Z_t}^- + \mathbf{b_t} \mathbf{Z_t}^+$,

  $\mathbf{b_t}$ = borrowing repo rate ; $\mathbf{l_t}$ = lending repo rate. (Brigo'16 et al, Bielecki-Rutkowski'15)
- **Effect of a large seller** on the default intensity
  (Dum.-Gri.-Quenez-S.'18), ...

The market $\mathcal{M}^f$ is **incomplete** :

2 sources of risk (W and M) ; only 1 risky asset for investment.
Indeed, let $\eta \in L^2(\mathcal{G}_T)$ be the terminal payoff of a European option.
It might not be possible to find (x, Z) in $\mathbb{R} \times \mathbb{H}^2$ such that

$$V_T^{x,Z} = \eta.$$

In other words, there does not necessarily exist $(V, Z) \in \mathbb{H}^2 \times \mathbb{H}^2$/

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_T = \eta.$$

$\exists!$ $(Y, Z, \mathbf{K})$ in $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ solution of the BSDE with default

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - \mathbf{K_t} dM_t; \quad Y_T = \eta \tag{2.1}$$

but in general $\mathbf{K} \neq Z\beta$.

American option superhedging

American option with maturity T and payoff $(\xi_t) \in \mathbb{S}^2$.

Here $\mathbb{S}^2$ is the vector space of $\mathbb{R}$-valued optional (not necessarily RCLL) processes ϕ such that $|||\phi|||^2_{\mathbb{S}^2} := \mathbb{E}[\text{ess sup}_{\tau \in \mathcal{T}} |\phi_\tau|^2] < \infty$.
Note that S^2 is the sub-space of RCLL processes of $\mathbb{S}^2$.

Definition

A superhedge for the seller against the American option with initial price $x \in \mathbb{R}$ is a strategy $Z \in \mathbb{H}^2$ s.t. $V_t^{x,Z} \geq \xi_t$, $0 \leq t \leq T$ a.s.

Let $\mathcal{A}(x) := \{$superhedges for the seller associated with $x\}$

Definition (Seller's (superhedging) price at time 0)

$$u_0 := \inf\{x \in \mathbb{R},\ \exists Z \in \mathcal{A}(x)\}.$$

Dual representation of the seller's price

The linear (incomplete) case with a RCLL payoff

Recall that in the case of the linear incomplete market, that is when $f(t, y, z) = -r_t y - \theta_t z$, we have the following dual representation of u_0 :

$$u_0 = \sup_{(R,\tau)\in\mathscr{P}\times\mathcal{T}} \mathbb{E}_R(e^{-\int_0^\tau r_s ds}\xi_\tau)$$

where $\mathscr{P}$ is the set of all martingale probability measures Kramkov '96.

We recall that a probability $R \sim P$ is a martingale probability measure if

- the discounted risky-asset price $(e^{-\int_0^t r_s ds} S_t)$ is a martingale under R
- $\Leftrightarrow$ for all $x \in \mathbb{R}, Z \in \mathbb{H}^2$, the discounted wealth process $(e^{-\int_0^t r_s ds} V_t^{x,Z})$ (where $V^{x,Z}$ follows the dynamics with the linear driver) is a martingale under R.

The nonlinear (incomplete) case with irregular payoff

Key tools for the dual representation

- Non-linear operator $\mathscr{E}_Q^f$
- a suitable set $\mathscr{Q}$ of equivalent probability measures (called f-martingale probability measures).
- $\mathscr{E}^f$-optional decomposition for strong $\mathscr{E}_Q^f$-supermartingales, $\forall Q \in \mathscr{Q}$

Non-linear f-expectation under Q : $\mathscr{E}^f_{\mathbf{Q}}$

Let Q be a probability measure equivalent to P.
From the $\mathbb{G}$-martingale representation thm, its density process (ζ_t) satisfies

$$d\zeta_t = \zeta_{t^-}(\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1,$$

where (α_t) and (ν_t) are $\mathbb{G}$-predictable processes with $\nu_{\vartheta \wedge T} > -1$ a.s.

By Girsanov's theorem, $W^Q_t := W_t - \int_0^t \alpha_s ds$ is a Q-Brownian motion, and $M^Q_t := M_t - \int_0^t \nu_s \lambda_s ds$ is a Q-martingale.

We have a representation for Q-martingales w.r.t. W^Q and M^Q.

f-expectation under Q : $\mathscr{E}^f_Q$

We call f-expectation under Q, the operator $\mathscr{E}^f_Q$ defined by :
for $\xi \in L^2_Q(\mathcal{G}_T)$,

$$\mathscr{E}^f_{Q,s,T}(\xi) := X_s, \qquad s \in [0,T]$$

where $(X,Z,K) \in S^2_Q \times \mathbb{H}^2_Q \times \mathbb{H}^2_{Q,\lambda}$ satisfies the Q-**BSDE**

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW^Q_t - K_t dM^Q_t; \quad X_T = \xi.$$

Definition

Let $Y \in \mathbb{S}^2_Q$. The process (Y_t) is said to be a strong $\mathscr{E}^f_Q$-supermartingale, if $\forall \sigma, \tau \in \mathcal{T}$ with $\sigma \leq \tau$,

$$\mathscr{E}^f_{Q,\sigma,\tau}(Y_\tau) \leq Y_\sigma \quad \text{a.s.}$$

In the case of equality, the process (Y_t) is said to be a (strong) $\mathscr{E}^f_Q$-martingale

Definition

A probability $Q \sim P$ is called an f-martingale probability measure if :
$\forall\, x \in \mathbb{R}$ and $\forall\, Z \in \mathbb{H}^2_Q$, the wealth $V^{x,Z}$ is a $\mathcal{E}^f_Q$-martingale.

We denote by $\mathcal{Q} := \{\ f\text{-martingale probability measures}\ \}$

- $\int(\sigma_t dW_t + \beta_t dM_t)$ is a Q-martingale $\forall Q \in \mathcal{Q}$
- $P \in \mathcal{Q}$

Theorem (Non-linear $\mathcal{E}^f$-optional decomposition G-Q-S-2018)

Let $(X_t) \in \cap_{Q \in \mathcal{Q}} \mathbb{S}^2_Q$ *(not necessarily RCLL). If* (X_t) *is a strong* $\mathcal{E}^f_{\mathbf{Q}}$*-supermartingale* $\forall\ \mathbf{Q} \in \mathcal{Q}$*, then there exists* $Z \in \mathbb{H}^2$, $\mathbf{C} \in \mathbb{C}^2$*, and a nondecreasing optional RCLL process* $\mathbf{h}$ *with* $h_0 = 0$ *and* $\mathbb{E}[h_T^2] < \infty$ *s.t.*

$$-dX_t = f(t, X_t, Z_t)dt - Z_t(dW_t + \beta_t dM_t) + d\mathbf{C}_{t-} + d\mathbf{h}_t.$$

This decomposition is unique.

Here $\mathbb{C}^2$ is the set of purely discontinuous non decreasing RCLL optional processes C with $C_{0^-} = 0$ and $\mathbb{E}(C_T^2) < \infty$.

Remark : $\mathbf{C}_t - \mathbf{C}_{t-} = -(X_{t+} - X_t)$.

Seller's price v_0 of a European option

Consider a European option with maturity T and payoff $\xi \in \mathbf{L^2}(\mathcal{G}_T)$.
Superhedging price v_0 :

$$v_0 := \inf\{x \in \mathbb{R} : \exists \varphi \in \mathbb{H}^2 \text{ s.t. } V_T^{x,\varphi} \geq \xi \text{ a.s.}\}.$$

Hyp : $\exists\, x \in \mathbb{R}$ and $\psi \in \mathbb{H}^2$ s.t. $\xi \leq V_T^{x,\psi}$ a.s. ($\Leftrightarrow v_0 < +\infty$).

Theorem (Pricing-hedging duality)

Suppose $\xi \in \cap_{Q \in \mathcal{Q}} L_Q^2$. The superhedging price v_0 of the European option satisfies

$$v_0 = \sup_{Q \in \mathcal{Q}} \mathcal{E}^f_{Q,0,T}(\xi).$$

Seller's price u_0 of the American option

Consider an American option with maturity T and payoff $(\xi_t) \in \mathbb{S}^2$. Assume that there exist $x \in \mathbb{R}$ and $\psi \in \mathbb{H}^2$ satisfying for all $0 \leq t \leq T$

$$\xi_t \leq V_t^{x,\psi}$$

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Theorem (Pricing-hedging duality)

Suppose $(\xi_t) \in \cap_{Q \in \mathcal{Q}} \mathbb{S}^2_Q$. The superhedging price u_0 of the American option satisfies

$$u_0 = \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}^f_{Q,0,\tau}(\xi_\tau).$$

Sketch of the proof

Recall that $u_0 := \inf\{x \in \mathbb{R},\ \exists Z \in \mathbb{H}^2 \text{ s.t. } V_.^{x,Z} \geq \xi_.\}$.

- Proof of the inequality :

$$u_0 \geq \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{Q,0,\tau}(\xi_\tau). \tag{3.1}$$

Let $x \in \mathbb{R}$ be such that $\exists Z \in \mathbb{H}^2$ with $V_.^{x,Z} \geq \xi_.$.
By definition, for each f-martingale probability measure Q, the wealth process $V_.^{x,Z}$ is an $\mathcal{E}_Q$-martingale. The result easily follows.
(*we have* $x = V_0^{x,Z} = \mathcal{E}_{Q,0,\tau}(V_\tau^{x,Z}) \geq \mathcal{E}_{Q,0,\tau}(\xi_\tau)$*. Hence,*
$x \geq \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{Q,0,\tau}(\xi_\tau)$*, which implies* (3.1))

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  (*we have* $x = V_0^{x,Z} = \mathcal{E}_{Q,0,\tau}(V_\tau^{x,Z}) \geq \mathcal{E}_{Q,0,\tau}(\xi_\tau)$. *Hence,* $x \geq \sup_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathcal{E}_{Q,0,\tau}(\xi_\tau)$, *which implies* (3.1))

- Proof of the converse inequality

To this aim, we introduce the dual value function at time $S \in \mathcal{T}$

$$Y(S) := \underset{Q \in \mathcal{Q}}{ess\sup}\ \underset{\tau \geq S}{ess\sup}\, \mathcal{E}_{Q,S,\tau}(\xi_\tau) \tag{3.2}$$

The study of this optimization problem is quite technical.
Let us first recall the simpler case when the market is complete.

The case of a complete market

Suppose $\beta = 0$ and the (Brownian) filtration is associated with W.
$\rightarrow$ the market is complete and $\mathcal{Q} = \{P\}$.
In this case, the dual value is given by the following *optimal stopping problem* with $\mathcal{E}^f$-expectation at time $S \in \mathcal{T}$

$$Y(S) := ess \sup_{\tau \geq S} \mathcal{E}^f_{P,S,\tau}(\xi_\tau) \tag{3.3}$$

For each $S \in \mathcal{T}$, $Y(S) = Y_S$ a.s., where Y is equal to the solution of the **reflected** BSDE with obstacle ξ and driver f (in the sense of Grig-I-O-Quen. (18)), that is $Y \in \mathbb{S}^2$ with $Y_t \geq \xi_t$ and $\exists!\ (Z,K) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$, a **predictable** process $A \in \mathcal{A}^2$ s.t.

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + \mathbf{dC}_{t^-}; \quad Y_T = \xi_T;$$

$$(Y_{t-} - \xi_{t-})(A_t - A_{t-}) = 0 \quad \text{(Skorokhod cond.)};$$

$$(Y_t - \xi_t)(\mathbf{C}_t - \mathbf{C}_{t-}) = 0 \quad \textbf{(Skorokhod cond.)}$$

The case of a complete market

Using this characterisation of the value of the above non-linear optimal stopping problem, we can show that the superhedging price u_0 of the American option satisfies

$$u_0 = Y_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^f_{P,0,\tau}(\xi_\tau).$$

Note that here, P is the (unique) f-martingale probability measure.

The value function in the incomplete case

In this case, the dual value function at time $S \in \mathcal{T}$ is given by

$$Y(S) := ess \sup_{Q \in \mathcal{Q}} ess \sup_{\tau \geq S} \mathcal{E}_{Q,S,\tau}(\xi_\tau)$$

The value function in the incomplete case

In this case, the dual value function at time $S \in \mathcal{T}$ is given by

$$Y(S) := \underset{Q \in \mathcal{Q}}{ess \sup} \; \underset{\tau \geq S}{ess \sup} \, \mathcal{E}_{Q,S,\tau}(\xi_\tau) = \underset{Q \in \mathcal{Q}}{ess \sup} \; Y_S^Q \quad \text{a.s.} \tag{3.4}$$

where for each $Q \in \mathcal{Q}$,

$$Y_S^Q = \underset{\tau \geq S}{ess \sup} \, \mathcal{E}_{Q,S,\tau}(\xi_\tau);$$

In other terms, Y_S^Q is the value function of the optimal stopping problem with $\mathcal{E}_Q$-expectation and with payoff $\xi_.$.
We know that the process Y^Q is equal to the solution of the **reflected** BSDE under Q associated with obstacle ξ and driver f. Hence, by (3.4), the dual value function $Y(S)$ is equal to the **ess. sup.** of a family **of reflected BSDEs.**

- We have $\mathbb{E}[\text{ess sup}_{S\in\mathcal{T}} Y(S)^2] < \infty$.
- There exists a right-u.s.c process $(Y_t) \in \mathbb{S}^2$, called the value process s.t. $\forall\, S \in \mathcal{T}$, $Y_S = Y(S)$ a.s.

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- Hence, Y admits an $\mathcal{E}^f$-optional decomposition :
$\exists Z \in \mathbb{H}^2$, $\exists \mathbf{C} \in \mathbb{C}^2$, and a nondecreasing optional process h s.t.

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)ds + \int_0^t Z_s(dW_s + \beta_s dM_s) - h_t - \mathbf{C}_{t^-}, 0 \leq t \leq T.$$

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- By a **forward** comparison theorem, $V_.^{Y_0,Z} \geq Y_.\ (\geq \xi_.)$
Hence Z is a **superhedging strategy** (associated with Y_0), which implies $Y_0 \geq u_0$ and hence
$Y_0 = u_0 = \sup_{Q\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathcal{E}_{Q,0,\tau}(\xi_\tau)$. **Q.E.D**

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- There exists a right-u.s.c process $(Y_t) \in \mathbb{S}^2$, called the value process s.t. $\forall\, S \in \mathcal{T}$, $Y_S = Y(S)$ a.s.
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- Hence, Y admits an $\mathcal{E}^f$-optional decomposition :
  $\exists Z \in \mathbb{H}^2$, $\exists \mathbf{C} \in \mathbb{C}^2$, and a nondecreasing optional process h s.t.

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)ds + \int_0^t Z_s(dW_s + \beta_s dM_s) - h_t - \mathbf{C}_{t^-}, 0 \leq t \leq T.$$

- By a **forward** comparison theorem, $V_\cdot^{Y_0,Z} \geq Y_\cdot \ (\geq \xi_\cdot)$
  Hence Z is a **superhedging strategy** (associated with Y_0), which implies $Y_0 \geq u_0$ and hence
  $Y_0 = u_0 = \sup_{Q\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathcal{E}_{Q,0,\tau}(\xi_\tau)$. **Q.E.D**
  The process $(h_t + \mathbf{C}_{t^-})$ *can be interpreted as the cumulative amount the seller withdraws from the hedging portfolio up to time* t.

Characterization of the seller's price via a constrained reflected BSDE

We first provide a **predictable** $\mathcal{E}^f$-decomposition for strong $\mathcal{E}^f_{\mathbf{Q}}$-supermartingale $\forall\ \mathbf{Q} \in \mathcal{Q}$.

Theorem

Let $(X_t) \in \cap_{Q\in\mathcal{Q}} \mathbb{S}^2_Q$ be a strong $\mathcal{E}^f_{\mathbf{Q}}$-supermartingale $\forall\ \mathbf{Q} \in \mathcal{Q}$. There exists a unique process $(Z, K, \mathbf{A}, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2 \times \mathbb{C}^2$ such that

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - K_t dM_t + d\mathbf{A}_t + dC_{t-}$$

$$\mathbf{A}_\cdot + \int_0^\cdot (K_s - \beta_s Z_s)\lambda_s ds \in \mathcal{A}^2$$

$$(K_t - \beta_t Z_t)\lambda_t \le 0,\ t \in [0, T],\ dP \otimes dt - \text{a.e.}$$

*where $\mathcal{A}^2$= {**predictable** RCLL non-decreasing processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < +\infty$}*

Remark : $\mathbf{h}_t = \mathbf{A}_t - \int_0^t (K_s - \beta_s Z_s) dM_s$.

European case

Theorem (European seller's price)

$$v_0 = X_0,$$

where (X_t) *is a (RCLL) supersolution of a* **constrained BSDE**, *that is* $X \in S^2$ *and* $\exists (Z, K) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ *and a* **predictable** *nondecreasing RCLL process* $\mathbf{A}$ *with* $\mathbf{A}_0 = 0$ *and* $\mathbb{E}(\mathbf{A}_T^2) < \infty$ *such that*

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - K_t dM_t + d\mathbf{A}_t; \quad X_T = \xi\,;$$

$$\mathbf{A}_. + \int_0^. (K_s - \beta_s Z_s)\lambda_s ds \quad \text{is nondecreasing} \quad \text{and}$$

$$(K_t - \beta_t Z_t)\lambda_t \leq 0,\ dP \otimes dt - \text{a.e.}\,;$$

and it is the minimal process which satisfies these properties.

Theorem (American seller's price)

$$u_0 = Y_0,$$

where (Y_t) is a supersolution of the constrained **reflected** *BSDE that is, $Y \in \mathbb{S}^2$ with $Y_t \geq \xi_t$ and*
$\exists!\ (Z, K) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$, *a* **predictable** *process $A \in \mathcal{A}^2$ and a purely discontinuous nondecreasing RCLL optional process $\mathbf{C} \in \mathbb{C}^2$ s.t.*

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + \mathbf{dC}_{t^-}; \quad Y_T = \xi_T;$$

$$A_\cdot + \int_0^\cdot (K_s - \beta_s Z_s)\lambda_s ds \quad \text{is nondecreasing}$$

$$K_\vartheta - \beta_\vartheta Z_\vartheta \leq 0 \text{ a.s.}$$

$$(Y_t - \xi_t)(\mathbf{C}_t - \mathbf{C}_{t-}) = 0 \quad \textbf{(Skorokhod cond.)}$$

and it is the minimal process which satisfies these properties.

Theorem (American seller's price)

$$u_0 = Y_0,$$

where (Y_t) *is a supersolution of the constrained* **reflected** *BSDE that is,*
$Y \in \mathbb{S}^2$ *with* $Y_t \geq \xi_t$ *and*
$\exists!\ (Z,K) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$, *a* **predictable** *process* $A \in \mathcal{A}^2$ *and a purely discontinuous nondecreasing RCLL optional process* $\mathbf{C} \in \mathbb{C}^2$ *s.t.*

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t + d\mathbf{C}_{t-}; \quad Y_T = \xi_T\ ;$$

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and it is the minimal process which satisfies these properties.

Comparing to the complete case, there are two additional constraints, and the process A does not necessarily satisfies the Skorokhod cond.

Proof

- We apply the $\mathscr{E}^f$-predictable decomposition theorem to the value process (Y_t). (Recall $Y_S = ess\sup_{Q\in\mathscr{Q},\tau\geq S}\mathscr{E}_{Q,S,\tau}(\xi_\tau)$.

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- It remains to show the Skorokhod cond. : $(Y_S - \xi_S)(\mathbf{C}_S - \mathbf{C}_{S-}) = 0$ for all $S \in \mathcal{T}$. (i.e. the process $\mathbf{C}$ increases only when $Y = \xi$).

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  Recall that $\mathbf{C}_S - \mathbf{C}_{S-} = -(Y_{S+} - Y_S)$.
  Hence, it is sufficient to show that $(Y_S - \xi_S)(Y_{S+} - Y_S) = 0$. To this aim, we introduce the "strict value" function at time $S \in \mathcal{T}$,

$$Y^+(S) = ess \sup_{Q\in\mathscr{Q},\tau>S} \mathscr{E}_{Q,S,\tau}(\xi_\tau).$$

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Lemma : For all $S \in \mathcal{T}$, $Y^+(S) = Y_{S^+}$ a.s. (i.e. the family $(Y^+(S)$ is aggregated by the process of right limits of the value process Y) and

$$Y_S = Y_{S^+} \vee \xi_S \quad \text{a.s.}$$

$\rightarrow (Y_S - \xi_S)(Y_{S+} - Y_S) = 0$ a.s. for all $S \in \mathcal{T}$. **Q.E.D.**

Buyer's price of the American option

Maximal price which allows the buyer to construct a portfolio strategy and an exercise time so that he is superhedged.

Definition

A superhedge for the buyer against the American option with initial price $x \in \mathbb{R}$ is a **pair** $(\tau, Z) \in \mathcal{T} \times \mathbb{H}^2$ s.t. $V_\tau^{-x,Z} + \xi_\tau \geq 0$ a.s.

Starting with amount $-x$ at $t = 0$ and following the strategy Z, the payoff received at τ allows the buyer to recover the debt incurred at $t = 0$ by buying the option.

Let $\mathcal{B}(x) := \{$superhedges for the buyer associated with $x\}$

Definition (buyer's (superhedging) price)

$$\tilde{u}_0 = \sup\{x \in \mathbb{R},\ \exists(\tau, Z) \in \mathcal{B}(x)\}.$$

Theorem

Suppose that there exist $x' \in \mathbb{R}$ and $\psi' \in \mathbb{H}^2$ satisfying $-\xi_t \leq V_t^{-x',\psi'}$, and that (ξ_t) is RCLL and left-u.s.c. along stopping times.
The buyer's price $\tilde{u}_0$ *satisfies*

$$\tilde{u}_0 = \inf_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \{-\mathcal{E}^f_{Q,0,\tau}(-\xi_\tau)\} = \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{Q}} \{-\mathcal{E}^f_{Q,0,\tau}(-\xi_\tau)\}.$$

Moreover, there exists a superhedge $(\tilde{\tau}, \tilde{Z})$ for the buyer.

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Moreover, there exists a superhedge $(\tilde{\tau}, \tilde{Z})$ *for the buyer.*

Idea of the proof : study of the dual (game) problem : for each $S \in \mathcal{T}$,

$$\tilde{Y}(S) = ess \inf_{Q \in \mathcal{Q}} ess \sup_{\tau \geq S} -\mathcal{E}_{Q,S,\tau}(-\xi_\tau)$$

We introduce $\tilde{\tau} := \inf\{t \in [0,T] : \tilde{Y}_t = \xi_t\}$ and prove there exists a portfolio strategy $\tilde{\varphi} \in \mathbb{H}^2$ such that $(\tilde{\tau}, \tilde{\varphi})$ is a superhedge for the buyer.
Remark : the result still holds when, instead of being RCLL on $[0,T]$*,* ξ *is only right-u.s.c., and RCLL only at* $\tilde{\tau}$.

In the proof, in order to study the properties of the dual value $\tilde{Y}$ (in particular a dynamic programming principle), we note that

$$\tilde{Y}(S) = ess \inf_{Q \in \mathcal{Q}} ess \sup_{\tau \geq S} -\mathcal{E}_{Q,S,\tau}(-\xi_\tau) = ess \inf_{Q \in \mathcal{Q}} \tilde{Y}_S^Q \quad \text{a.s.} \qquad (5.1)$$

where, for each $Q \in \mathcal{Q}$,

$$\tilde{Y}_S^Q = ess \sup_{\tau \in \mathcal{T}} -\mathcal{E}_{Q,S,\tau}(-\xi_\tau)$$

Now, $\tilde{Y}^Q$ can be shown to be equal to the solution of a **reflected** BSDE under Q with lower obstacle ξ. (associated with an appropriate driver $\tilde{f}$). Hence, by (5.1), the dual value $\tilde{Y}(S)$ is equal to the **ess inf** of a family of **reflected BSDEs.**

Contrary to the seller's case (for which we had an ess. supremum), we have here an **ess. infimum**, which leads to some specific technical problems.

Buyer's nearly superhedging price

When ξ does not satisfy any regularity assumption on the left, the **value of this game** can be interpreted as a **nearly superhedging price**, defined as the supremum of the initial prices which, for each $\varepsilon > 0$, allow the buyer to be ε–superhedged, that is

$$\bar{u}_0 = \sup\{z \in \mathbb{R},\ \forall \varepsilon > 0, \exists (\tau, \varphi) \in \mathcal{T} \times \mathbb{H}^2 \text{ s.t. } V_\tau^{-z,\varphi} \geq -\xi_\tau - \varepsilon \text{ a.s.}\}.$$

$\bar{u}_0 \geq \tilde{u}_0$ (the buyer's superhedging price) (we may have $\tilde{u}_0 = -\infty$).

No arbitrage considerations

Definition

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$ and φ in $\mathbb{H}^2$. We say that (y, φ) is an *arbitrage opportunity for the seller* of the American option with initial price x if

$$y < x \quad \text{and} \quad V_\tau^{y,\varphi} - \xi_\tau \geq 0 \quad \text{a.s.} \quad \text{for all } \tau \in \mathcal{T}.$$

Definition

Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$, $\tau \in \mathcal{T}$ and $\varphi \in \mathbb{H}^2$. We say that (y, τ, φ) is an *arbitrage opportunity for the buyer* of the American option with initial price x if

$$y > x \quad \text{and} \quad V_\tau^{-y,\varphi} + \xi_\tau \geq 0 \quad \text{a.s.}$$

Proposition : Let $x \in \mathbb{R}$. There exists an arbitrage opportunity for the seller (resp. for the buyer) of the American option with price x if and only if $x > u_0$ (resp. $x < \tilde{u}_0$).

Definition

A real number x is called an *arbitrage-free price for the American option* if there exists no arbitrage opportunity, neither for the seller nor for the buyer.

Proposition : If $u_0 < \tilde{u}_0$, there does not exist any arbitrage-free price for the American option. If $u_0 \geq \tilde{u}_0$, the interval $[\tilde{u}_0, u_0]$ is the set of all arbitrage-free prices. We call it the *arbitrage-free* interval for the American option.

Conclusion

Key results

In a non-linear incomplete market with default, we have given

- a **dual representation of the seller's superhedging price** of the American option (with completely irregular payoff) in terms of the value of a non-linear mixed control/stopping problem. The dual representation involves a suitable set of equivalent probability measures $Q \in \mathcal{Q}$, which we call f-martingale probability measures.
- Characterization of the seller's superhedging price as the **minimal supersolution of a constrained reflected BSDE** with default.
- a **duality representation for the buyer's price** in terms of the value of a non-linear control/stopping **game** problem.

The talk is based on these 2 papers :

- Grigorova M., Quenez M.-C. and A. Sulem : European options in a non-linear incomplete market with default, *SIAM Journal on Financial Mathematics*, 11(3), (2020), 849-880.
- Grigorova M., Quenez M.-C. and A. Sulem : American options in a non-linear incomplete market model with default, https ://hal.archives-ouvertes.fr/hal-02025835/document

The complete case with defaultable asset with total default is treated in :

- Dumitrescu R., Quenez M-C., and A. Sulem , American options in an imperfect complete market with default, *ESAIM Proceedings & Surveys*, Vol 64, pp 93–110, 2018.

General study of BSDEs with default jump can be found in :

- Dumitrescu D., Grigorova M., Quenez M.-C. and A. Sulem : BSDE with default jump, (2019), Computation and Combinatorics in Dynamics, Stochastics and Control ; Abel Symposium 2016, 13, Springer.