Dupire-Itô's formula for $\mathbb{C}^{0,1}$ or concave path-dependent functionals and applications

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Based on works with P. Cardaliaguet (Dauphine-PSL), G. Loeper (Monash Univ. and BNP), X. Tan (Chinese University of Hong Kong)

Motivation

☐ Let $v : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ be $C^{1,2}$ and let $X = M + A$ be a continuous semi-martingale, then

$$v(t, X_t) = v(0, X_0) + \int_0^t Dv(s, X_s) dM_s + \Gamma_t, \qquad (1)$$

in which

$$\Gamma_t := \int_0^t \partial_t v(s, X_s) ds + \int_0^t Dv(s, X_s) dA_s + \frac{1}{2} \int_0^t D^2 v(s, X_s) d[X]_s.$$

☐ Still holds when v is juste $C^{0,1}$ (or concave non-increasing in time) for some Γ.

☐ Can we extend this result in the case of path-dependent functionals using the notion of Dupire's derivative ?
⇒ **Dupire's differentiability is difficult to prove ! Requiring juste $C^{0,1}$ would help.**

Dupire-Itô's formula

Definitions

- Spaces and norm
  - $C([0,T])/D([0,T])$: space of continuous/càdlàg paths on $[0,T]$
  - $\Theta := [0,T] \times D([0,T])$.
  - $\|\mathrm{x}\| := \sup_{s\in[0,T]} |\mathrm{x}_s|$,
  - $\mathrm{x}_{t\wedge} := (\mathrm{x}_{t\wedge s})_{s\in[0,T]}$, (optional) stopped path.
  - $\mathrm{v} : D([0,T]) \to \mathbb{R}$ is non-anticipative.
- Horizontal derivative :

$$\partial_t \mathrm{v}(t,\mathrm{x}) \;:=\; \lim_{h\searrow 0} \frac{\mathrm{v}(t+h,\mathrm{x}_{t\wedge}) - \mathrm{v}(t,\mathrm{x})}{h}$$

- Vertical derivative :

$$\nabla_{\mathrm{x}} \mathrm{v}(t,\mathrm{x}) \;:=\; \lim_{y\to 0} \frac{\mathrm{v}(t,\mathrm{x}\oplus_t y) - \mathrm{v}(t,\mathrm{x})}{y},$$

where $\mathrm{x}\oplus_t y \;:=\; \mathrm{x}\mathbf{1}_{[0,t)} + (\mathrm{x}_t + y)\mathbf{1}_{[t,T]}$.

- Regularity
  - $\mathrm{v} \in \mathbb{C}(\Theta)$ if is continuous.
  - $\mathrm{v} \in \mathbb{C}_l(\Theta)$ if $\forall\ (t, \mathrm{x}) \in \Theta$ and $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$t' \leq t,\ |t - t'| + \|\mathrm{x}_{t\wedge} - \mathrm{x}'_{t'\wedge}\| \leq \delta \implies |\mathrm{v}(t, \mathrm{x}) - \mathrm{v}(t', \mathrm{x}')| \leq \varepsilon.$$

  - $\mathbb{C}^{0,1}(\Theta)$: $\mathrm{v} \in \mathbb{C}(\Theta)$ and $\nabla_{\mathrm{x}} \mathrm{v} \in \mathbb{C}_l(\Theta)$.
  - $\mathbb{C}^{1,2}(\Theta)$: $\mathrm{v} \in \mathbb{C}^{0,1}(\Theta)$, $\partial_t \mathrm{v}$ and $\nabla^2_{\mathrm{x}} \mathrm{v}$ belong to $\mathbb{C}_l(\Theta)$.

Dupire-Itô's formula

□ Assume that $\mathrm{v} \in \mathbb{C}^{1,2}(\Theta)$ and that $X = M + A$ is a continuous semi-martingale, then

$$\mathrm{v}(t, X_t) \quad = \quad \mathrm{v}(0, X_0) + \int_0^t \nabla_{\mathrm{x}} \mathrm{v}(s, X_s) dM_s + \Gamma_t,$$

in which

$$\Gamma_t := \int_0^t \partial_t \mathrm{v}(s, X_s) ds + \int_0^t \nabla_{\mathrm{x}} \mathrm{v}(s, X_s) dA_s + \frac{1}{2} \int_0^t \nabla_{\mathrm{x}}^2 \mathrm{v}(s, X_s) d[X]_s.$$

See Dupire [6] and Cont and Fournié [4].

□ Remark : See also Saporito [4] for a Meyer-Tanaka type formula assuming $\mathbb{C}^1$-regularity in time.

The case of $\mathbb{C}^{0,1}$, concave and non-increasing in time functionals

Simple but very helpful !

Based on B. and Tan [2].

□ One can expect to keep the decomposition because the monotonicity in time + concavity just gives an additional monotone term.

□ We say that v is **non-increasing in time** if $\mathrm{v}(t+h, \mathrm{x}_{\wedge t}) - \mathrm{v}(t, \mathrm{x}) \leq 0$ when $h \geq 0$. We say that v is **Dupire-concave** if for $\mathrm{x}^1 = \mathrm{x}^2$ on $[0, t)$

$$\mathrm{v}(t, \theta \mathrm{x}^1 + (1-\theta)\mathrm{x}^2) \;\geq\; \theta \mathrm{v}(t, \mathrm{x}^1) + (1-\theta)\mathrm{v}(t, \mathrm{x}^2), \;\; \text{for all } \theta \in [0,1]$$

□ Simple argument : Set $X^n := \sum_i X_{t_i^n} \mathbf{1}_{[t_i^n, t_{i+1}^n)}$. Then,

$$\mathrm{v}(t_{i+1}^n, X^n) - \mathrm{v}(t_i^n, X^n) = \mathrm{v}(t_{i+1}^n, X^n) - \mathrm{v}(t_{i+1}^n, X^n_{\wedge t_i^n}) + \underbrace{\mathrm{v}(t_{i+1}^n, X^n_{\wedge t_i^n}) - \mathrm{v}(t_i^n, X^n)}_{\leq 0}.$$

Applying Meyer-Tanaka to $r \mapsto \mathrm{v}(t_{i+1}^n, X^n_{\wedge t_i^n} \oplus_{t_{i+1}^n} (X_r - X_{t_i^n}))$,

$$\begin{aligned} &\mathrm{v}(t_{i+1}^n, X^n) - \mathrm{v}(t_{i+1}^n, X^n_{\wedge t_i^n}) \\ &= \int_{t_i^n}^{t_{i+1}^n} \nabla_{\mathrm{x}} \mathrm{v}(t_{i+1}^n, X^n_{\wedge t_i^n} \oplus_{t_{i+1}^n} (X_r - X_{t_i^n})) dX_r + K^n_{t_{i+1}^n} - K^n_{t_i^n} \end{aligned}$$

in which K^n is a non-increasing predictable process starting at 0.

□ **Proposition :** Let X be a continuous semi-martingale, $\mathrm{v} \in \mathbb{C}^{0,1}(\Theta)$ with $\nabla_{\mathrm{x}}\mathrm{v}$ locally bounded. Assume that there exists $R \in \mathbb{C}^{1,2}(\Theta)$ and a continuous function $\ell : [0, T] \to \mathbb{R}$ such that :

(1) $\mathrm{v} - R$ is Dupire-concave.

(2) $s \mapsto \mathrm{v}(s, \mathrm{x}_{t\wedge \cdot}) - \ell(s)$ is non-increasing on $[t, T]$, for any $(t, \mathrm{x}) \in \Theta$.

Then, there exists predictable process Γ starting at 0 such that

$$\mathrm{v}(\cdot, X) = \mathrm{v}(0, X) + \int_0^{\cdot} \nabla_{\mathrm{x}}\mathrm{v}(r, X) dX_r + \Gamma.$$

Application to second order BSDE's

Motivation : dual formulation for a hedging problem with price impact, see B. and Tan [2].

Primal problem : A probability measure $\mathbb{P}$, together with

two constants $y_0, v_0 \in \mathbb{R}$ and predictable processes $\big(Y, V, \mathfrak{B}, \mathfrak{g}\big)$,

(i) X is a $\mathbb{P}$-martingale such that $\mathbb{P}[X_0 = x_0] = 1$, $d\langle X\rangle_t = \sigma_t^2(X, \mathfrak{g}_t)dt$ on $[0, T]$, and $\mathfrak{g} = (\mathfrak{g}_t)_{0\le t\le T}$ satisfying

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T |\sigma_t(X, \mathfrak{g}_t)|^2 dt\Big] < \infty \;\text{ and }\; \int_0^T |\mathfrak{g}_t|^2 d\langle X\rangle_t < \infty,$$

(ii) $\mathfrak{B}$ with bounded total variation, and (Y, V) satisfy : $V_T = \Phi(X)$, $\mathbb{P}$-a.s,

$$V_t = v_0 + \int_0^t F_s(X, \mathfrak{g}_s)ds + \int_0^t Y_s dX_s \text{ and } Y_t = y_0 + \int_0^t \mathfrak{g}_s dX_s - \mathfrak{B}_t,\; t \in [0, T],$$

□ **Assumption :** there exists $G : [0,T] \times D([0,T]) \times [0,\infty) \to \mathbb{R}$, such that, for all $(t,\mathrm{x}) \in [0,T] \times D([0,T])$ and $a > 0$,

$$G_t(\mathrm{x}, a) = F_t(\mathrm{x}, \sigma_t^{-1}(\mathrm{x}, a)) \ \text{ and } \ \partial_a G_t(\mathrm{x}, a) = a\sigma_t^{-1}(\mathrm{x}, a).$$

(+ Fréchet differentiability of Φ and ...)

□ **Theorem :** If there exists a (weak-)solution to the dual problem

$$\mathrm{v}(t,\mathrm{x}) \ := \sup_{\alpha} \mathbb{E}\Big[\Phi\big(\bar{X}^{t,\mathrm{x},\alpha}\big) - \int_t^T G_s\big(\bar{X}^{t,\mathrm{x},\alpha}, \alpha_s\big) ds\Big],$$

where

$$\bar{X}^{t,\mathrm{x},\alpha} = \mathrm{x}_{\wedge t} + \int_t^{\cdot} \alpha_s dW_s,$$

then there exists a solution to the primal problem.

Proof :

1. Use calculus of variations to characterize the optimal $\hat{\alpha}$.
2. Show that v is Dupire-concave, non-increasing in time (up to a smooth term), and $\mathbb{C}^{0,1}(\Theta)$.
3. Apply our version of Dupire–Itô's Lemma and check that this provides a solution to the primal problem by using 1.

Little hope for the functional to be $\mathbb{C}^{1,2}$...

□ Rem : we provide sufficient conditions for existence in the dual problem.

Without $\mathbb{C}^{0,1}$-regularity

Slightly more difficult, removing the differentiability.

Based on B. and Tan [3].

□ If v is **Dupire-concave**, one can define its super-differential

$$\partial \mathrm{v}(t,\mathrm{x}) \;:=\; \big\{z : \mathrm{v}(t, \mathrm{x} \oplus_t y) \le \mathrm{v}(t,\mathrm{x}) + z \cdot y,\ \forall\, y\big\}.$$

□ **Right equi-continuous :** $\mathrm{v} \in \mathbb{C}^{\mathrm{e}}_{\mathrm{r}}(\Theta)$, if $\forall\ (t,\mathrm{x}) \in \Theta$ and $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$t' \ge t,\ |t'-t| + \|\mathrm{x}'_{t'\wedge\cdot} - \mathrm{x}_{t\wedge\cdot}\| \le \delta \implies \big|\mathrm{v}(t', \mathrm{x}' \boxplus_{t'} y) - \mathrm{v}(t, \mathrm{x} \boxplus_t y)\big| \le \varepsilon,$$

$\forall\ |\mathrm{x}_t - y| \le 1$. Here $\mathrm{x} \boxplus_t y := \mathrm{x}\mathbf{1}_{[0,t)} + y\mathbf{1}_{[t,T]}$.

□ **Locally equi-nonincreasing in time :** $\forall\ K > 0$, $\exists$ a non-decreasing function $\mathrm{r}_K : [0, T] \to \mathbb{R}$ and a module continuity $\rho_K : \mathbb{R}_+ \to \mathbb{R}_+$ s.t., $\forall$ $0 \le t \le t+h \le T$, $\|\mathrm{x}\| \le K$, and $|\mathrm{x}_t - y| \le 1$,

$$\mathrm{v}(t+h, \mathrm{x}_{t\wedge\cdot} \boxplus_{t+h} y) \;\le\; \mathrm{v}(t, \mathrm{x} \boxplus_t y) \;+\; \rho_K(|y - \mathrm{x}_t|)\big(\mathrm{r}_K(t+h) - \mathrm{r}_K(t)\big).$$

Robust optional decomposition

□ Let $\mathcal{P}$ denote the collection of all Borel probability measures on $D([0,T])$, under which X is a càdlàg semimartingale.

□ **Theorem** Let $\mathrm{v} \in \mathbb{C}^{\mathrm{e}}_{\mathrm{r}}(\Theta)$ be Dupire-concave, locally equi-nonincreasing in time, and such that

$$\sup\left\{|\mathrm{v}(t,\mathrm{x})|+|z| \ : (t,\mathrm{x}) \in \Theta,\ \|\mathrm{x}\| \leq K,\ z \in \partial \mathrm{v}(t,\mathrm{x})\right\} < \infty, \ \text{ for all } K > 0.$$

Then, there exists a predictable locally bounded process $H : \Theta \to \mathbb{R}$, together with a collection of non-decreasing processes $\{C^{\mathbb{P}} : \mathbb{P} \in \mathcal{P}\}$, satisfying

$$\mathrm{v}(t,X) = \mathrm{v}(0,X) + \int_0^t H_s dX_s - C^{\mathbb{P}}_t,\ t \in [0,T],\ \mathbb{P}\text{−a.s.}\ \forall\, \mathbb{P} \in \mathcal{P}.$$

Moreover, $H_s \in \partial \mathrm{v}(s, X^{s-})$ for all $s \in [0,T]$, $\mathcal{P}$-q.s, where

$$X^{s-}_t := X_t \mathbf{1}_{t\in[0,s)} + X_{s-}\mathbf{1}_{t\in[s,T]}.$$

Proof : Smooth-out the vertical derivative and pass to the limit

Remark : One actually shows that

$$H_s = \int \partial^+ \mathrm{v}(s, X^{s-}; y)(-\nabla\phi(y))dy$$

where ϕ is the smoothing kernel and

$$\begin{aligned} \partial^+ \mathrm{v}(r, \mathrm{x}; y) &:= \lim_{\varepsilon \searrow 0} \frac{\mathrm{v}\big(r, \mathrm{x} \oplus_r (\mathrm{x}_r + \varepsilon y)\big) - \mathrm{v}(t, \mathrm{x})}{\varepsilon} \\ &= \min\{y \cdot z : z \in \partial \mathrm{v}(r, \mathrm{x})\}. \end{aligned}$$

Application to robust super-hedging

$\mathcal{M}^+(t,\mathrm{x}) := \{\mathbb{Q} \text{ on } D([0,T]) \ : \mathbb{Q}[X_{t\wedge\cdot} = \mathrm{x}_{t\wedge\cdot}] = 1,\ X \geq 0 \text{ is } \mathbb{Q}\text{–martingale}\}.$

Given

$$M_t(\mathrm{x}) := \sup_{0\leq s\leq t} \mathrm{x}_s, \quad m_t(\mathrm{x}) := \inf_{0\leq s\leq t} \mathrm{x}_s, \quad A_t(\mathrm{x}) := \int_0^t \mathrm{x}_s \mu(ds),$$

where μ is a finite signed measure on $[0,T]$ finitely many atoms. We fix a uniformly continuous function $\phi : \mathbb{R}^4 \to \mathbb{R}$,

$$\Phi(\mathrm{x}) \ := \ \phi\big(M_T(\mathrm{x}), m_T(\mathrm{x}), A_T(\mathrm{x}), \mathrm{x}_T\big),$$

such that

$$\big|\Phi(\mathrm{x})\big| \ \leq \ K\Big(1 + \mathrm{x}_T + \int_0^T \mathrm{x}_t |\mu|(dt)\Big), \quad \text{for all } \mathrm{x} \in D([0,T]),$$

and, for all $0 \leq M_0 \leq M_1$, $0 \leq m_1 \leq w_1 \wedge \varepsilon$ and $a_0, a_1 \in \mathbb{R}$,

$$\Big|\phi(M_1, m_1, a_1, w_1) - \phi(M_0, 0, a_0, 0)\Big| \leq K\big(|a_1 - a_0| + w_1\big).$$

□ **Theorem :** Let $\mathcal{A}$ be the collection of all locally bounded $\mathbb{F}$-predictable processes H such that $\int_0^{\cdot} H_r dX_r$ is $\mathbb{Q}$-a.s. bounded from below by a $\mathbb{Q}$-martingale, for all $\mathbb{Q} \in \mathcal{M}^+(0, x)$. Then,

$$\mathrm{v}(0,\mathrm{x}) := \sup_{\mathbb{Q}\in\mathcal{M}^+(0,\mathrm{x})} \mathbb{E}^{\mathbb{Q}}\big[\Phi(X)\big]$$

$$= \inf\Big\{ v \in \mathbb{R} \ : \exists H \in \mathcal{A} \text{ s.t. } v + \int_0^T H_r dX_r \geq \Phi(X),\ \mathcal{M}^+(0,x) - \mathrm{q.s.}\Big\}$$

Moreover the infimum is achieved by $H \in \partial \mathrm{v}(\cdot, X^{\cdot -})$.

□ Remark : Similar to Guo, Tan and Touzi [1] but under much restrictive continuity assumptions + explicit representation of H. Comparing with other works, we allow for jumps without any dominating assumption (compare with Nutz [2]).

$\mathbb{C}^{0,1}$ without concavity and time-monotonicity

Treat the robust hedging problem in “non-degenerate” cases,
e.g. bounded uncertain volatility.

Based on B., Loeper and Tan [1].

Weak Dirichlet processes

□ In the Markovian case : works by Russo and his co-authors, using the concept of weak Dirichlet processes and the stochastic calculus by regularization. See in particular Gozzi and Russo [8].

Definitions :

- Let X and Y be two real valued càdlàg processes. The co-quadractic variation $[X, Y]$ is defined by

$$[X, Y]_t \ := \ \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds,$$

  whenever the limit exists in the sense of u.c.p.
- X has finite quadratic variation, if $[X] := [X, X]$, exists and is finite a.s.
- A is **orthogonal** if $[A, N] = 0$ for any real valued continuous local martingale N.
- X is a **weak Dirichlet process** if $X = X_0 + M + A$, where M is a local martingale and A is orthogonal such that $M_0 = A_0 = 0$.

$\mathbb{C}^{0,1}$-Dupire-Itô's formula

□ We say that φ is **locally uniformly continuous** if, for each $K > 0$, there exists a modulus of continuity δ_K such that, for all $t \in [0, T]$, $h \in [0, T - t]$, $\|\mathrm{x}\| \leq K$, $|y| \leq K$,

$$\left|\varphi(t, \mathrm{x}) - \varphi(t + h, \mathrm{x}_{t\wedge})\right| + \left|\varphi(t, \mathrm{x}) - \varphi(t, \mathrm{x} \oplus_t y)\right| \;\leq\; \delta_K(h + |y|).$$

□ **Theorem :** Let $X = M + A$ be a continuous weak Dirichlet process with finite quadratic variation, $\mathrm{v} \in \mathbb{C}^{0,1}(\Theta)$ be such that v and $\nabla_{\mathrm{x}}\mathrm{v}$ are locally uniformly continuous, and $s \mapsto \nabla_{\mathrm{x}}\mathrm{v}(s, X)$ admits right-limits a.s.. Then,

$$\mathrm{v}(t, X) \;=\; \mathrm{v}(0, X) \;+\; \int_0^t \nabla_{\mathrm{x}}\mathrm{v}(s, X) dM_s \;+\; \Gamma_t, \quad t \in [0, T],$$

where Γ is a continuous orthogonal process **if and only if**

$$\frac{1}{\varepsilon}\int_0^{\cdot}\Big(\mathrm{v}(s + \varepsilon, X) - \mathrm{v}(s + \varepsilon, X_{s\wedge} \boxplus_{s+\varepsilon} X_{s+\varepsilon})\Big)\Big(N_{s+\varepsilon} - N_s\Big) ds \longrightarrow 0, \text{ u.c.p.} \qquad (2)$$

Sufficient condition

□ The condition (2) holds if $\mathrm{v} \in \mathbb{C}^{1,2}$.

□ It also holds if v is Fréchet differentiable or more generally if, for all $\mathrm{x} \in D([0,T])$, $s \in [0,T]$ and $\varepsilon \in [0, T-s]$,

$$\left|\mathrm{v}(s+\varepsilon, \mathrm{x}) - \mathrm{v}(s+\varepsilon, \mathrm{x}_{s\wedge} \oplus_{s+\varepsilon} (\mathrm{x}_{s+\varepsilon} - \mathrm{x}_s))\right| \leq \int_{(s,s+\varepsilon)} \phi\big(\mathrm{x}, |\mathrm{x}_u - \mathrm{x}_s|\big)\, db_u(\mathrm{x}),$$

where $\phi : C([0,T]) \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ satisfies $\sup_{|y| \leq K} \phi(\mathrm{x}, y) < \infty$, $\lim_{y \searrow 0} \phi(\mathrm{x}, y) = \phi(\mathrm{x}, 0) = 0$ for all $\mathrm{x} \in C([0,T])$ and $K > 0$, and $b : C([0,T]) \longrightarrow \mathrm{BV}_+$.

Application to robust hedging

□ Let us consider a payoff function of the form

$$g(X) \;=\; g_\circ\Big(\int_0^T X_t \mu(dt)\Big),$$

where $g_\circ \in C^{1+\alpha}(\mathbb{R})$ is bounded, and μ is a finite positive measure with at most finitely many atoms on $[0, T]$.

□ Let $\mathcal{P}_0$ be the collection of all probability measures $\mathbb{P}$ such that $\mathbb{P}[X_0 = x_0] = 1$ and

$$dX_s = \sigma_s dW_s^{\mathbb{P}},\; \sigma_s \in [\underline{\sigma}, \overline{\sigma}],\; s \in [0, T],\; \mathbb{P}\text{-a.s.} \qquad (3)$$

for some $\mathbb{P}$–Brownian motion $W^{\mathbb{P}}$.

□ Define the control problem :

$$\mathrm{v}(t, \mathrm{x}) \;:=\; \sup_{\mathbb{P}\in\mathcal{P}(t,\mathrm{x})} \mathbb{E}^{\mathbb{P}}\big[g(X)\big]$$

where

$$\mathcal{P}(t, \mathrm{x}) := \big\{\mathbb{P}\; : \mathbb{P}[X_{t\wedge} = \mathrm{x}_{t\wedge}] = 1,\; \text{and (3) holds on } [t, T]\big\}.$$

□ **Proposition :** v admits a vertical derivative and

$$|\mathrm{v}(t+h,\mathrm{x}_{t\wedge})-\mathrm{v}(t,\mathrm{x})|\le \overline{\sigma}h^{\frac{1}{2}}\mu([t,T])$$

$$|\mathrm{v}(t,\mathrm{x}')-\mathrm{v}(t,\mathrm{x})|\le \int_0^t|\mathrm{x}'_s-\mathrm{x}_s|\mu(ds).$$

$$|\nabla_{\mathrm{x}}\mathrm{v}(t,\mathrm{x}')-\nabla_{\mathrm{x}}\mathrm{v}(t,\mathrm{x})| \;\le\; C\Big(\Big|\int_0^t|\mathrm{x}'_s-\mathrm{x}_s|\mu(ds)\Big|^{\alpha}+|\mathrm{x}'_t-\mathrm{x}_t|^{\alpha}\Big),$$

$$|\nabla_{\mathrm{x}}\mathrm{v}(t',\mathrm{x}_{t\wedge})-\nabla_{\mathrm{x}}\mathrm{v}(t,\mathrm{x})| \;\le\; C\big(|t'-t|^{\frac{\alpha}{2+2\alpha}}+\mu([t,t'))\big),$$

□ Apply the $\mathbb{C}^{0,1}$-Dupire-Itô's Lemma : the **expected duality holds**, and that the **super-hedging strategy is given by $\nabla_{\mathrm{x}}\mathrm{v}$**.

□ The proof is based on a step-constant approximation of the path at the level of the PDE : $g(X)\to g(\sum_{i=1}^{n-1}X_{t_i^n}\mathbf{1}_{[t_i^n,t_{i+1}^n)}+X_{t_{n-1}^n}\mathbf{1}_{\{T\}})$

It applies more generally to equations of the form

$$\partial_t\mathrm{v}+H(\nabla^2_{\mathrm{x}}\mathrm{v})=0$$

with H convexe (with linear growth at infinity).

Towards a new approach to PPDE

In progress with Cardaliaguet, Loeper and Tan.

Definition of solutions by approximation

☐ Let $\pi = (\pi^n)_n$, with $\pi^n = (t_i^n)_{0\leq i\leq n}$, be an increasing sequence of time grids. Set

$$\bar{\mathrm{x}}^n := \sum_{i=0}^{n-1} \mathrm{x}_{t_i^n} 1_{[t_i^n, t_{i+1}^n)} + \mathrm{x}_{t_{n-1}^n} \mathbf{1}_{\{T\}}$$

☐ We say that a continuous function v^n is a π^n-**viscosity solution** of

$$-\partial_t\varphi(t,\mathrm{x}) - F(t,\mathrm{x},\varphi(t,\mathrm{x}),\nabla_{\mathrm{x}}\varphi(t,\mathrm{x}),\nabla^2_{\mathrm{x}}\varphi(t,\mathrm{x})) = 0 \ \forall\ t < T$$

if it is of the form

$$\sum_{i=0}^{n-1} \mathbf{1}_{[t_i^n, t_{i+1}^n)} v_i^n(t, \bar{\mathrm{x}}^n_{\wedge t_i^n}, x)$$

in which each v_i^n is a viscosity solution on $[t_i^n, t_{i+1}^n)$ of

$$-\partial_t v_i^n(t,\bar{\mathrm{x}}^n_{\wedge t_i^n},x) - F(t,\bar{\mathrm{x}}^n_{\wedge t_i^n}, v_i^n(t,\bar{\mathrm{x}}^n_{\wedge t_i^n},x), Dv_i^n(t,\bar{\mathrm{x}}^n_{\wedge t_i^n},x), D^2v_i^n(t,\bar{\mathrm{x}}^n_{\wedge t_i^n},x)) = 0$$

$$v_i^n(t_{i+1}^n-,\bar{\mathrm{x}}^n_{\wedge t_i^n},x) = v_{i+1}^n(t_{i+1}^n,\bar{\mathrm{x}}^n_{\wedge t_i^n}\boxplus_{t_{i+1}^n} x, x)$$

□ We say that v is a **π-approximate-viscosity solution** on $C([0,T])$ (resp. $D([0,T])$) of

$$-\partial_t \mathrm{v}(t,\mathrm{x}) - F(t,\mathrm{x},\mathrm{v}(t,\mathrm{x}),\nabla_{\mathrm{x}}\mathrm{v}(t,\mathrm{x}),\nabla^2_{\mathrm{x}}\mathrm{v}(t,\mathrm{x})) = 0\ ,\ t < T$$

with terminal condition

$$\mathrm{v}(T,\cdot) = g$$

if there exists a subsequence $(n_k)_{k\geq 1}$ such that $v^{n_k}(t,\mathrm{x},\mathrm{x}_t) \to \mathrm{v}(t,\mathrm{x})$ for all $(t,\mathrm{x}) \in [0,T] \times C([0,T])$ (resp. $[0,T] \times D([0,T])$) where $(v^n)_n$ is the sequence defined as above with

$$v^n(t^n_{n-1},\mathrm{x},x) = g(\bar{\mathrm{x}}^n \boxplus_{t^n_{n-1}} x)$$

□ Typical examples :

- Semi-linear PDEs : since they can be represented by BSDEs.
- HJB equations : whenever replacing X by $\bar{X}^n$ in the coefficients and payoff provides an approximation.

Existence, comparison, etc.

□ We focus on the case where

$$F(t, \mathrm{x}, r, p, q) = H(t, \mathrm{x}, r, p, q) + \rho(t, \mathrm{x})r + b(t, \mathrm{x})p + \frac{1}{2}\sigma^2(t, \mathrm{x})q$$

where all the coefficients are continuous, bounded, and Lipschitz (can relax to e.g. uniformly continuous in space,...).

□ **Theorem :** Let g be Lipschitz, then $\exists$ a unique π-approximate viscosity solution v on $C([0, T])$ with terminal condition g. Moreover,

- There exists $C > 0$ such that for all $K > 0$

$$\begin{cases} |\mathrm{v}(t, \mathrm{x}') - \mathrm{v}(t, \mathrm{x})| \le C(1+K)(\|\mathrm{x} - \mathrm{x}'\| + |\mathrm{x}_t - \mathrm{x}'_t|) \\ |\mathrm{v}(t', \mathrm{x}_{\wedge t}) - \mathrm{v}(t, \mathrm{x})| \le C(1+K)|t' - t|^{\frac{1}{2}} \\ |v^n(\cdot, \mathrm{x}, \mathrm{x}_\cdot) - \mathrm{v}(\cdot, \mathrm{x})| \le C\left(\max_{j<n} |t^n_{j+1} - t^n_j|^{\frac{1}{3}} + (1+K)\|\bar{\mathrm{x}}^n - \mathrm{x}\|\right) \end{cases}$$

  when $\|\mathrm{x}\| \vee \|\mathrm{x}'\| \le K$, and $t \le t' \in [0, T]$.

- If π' is another increasing sequence of time grids and if v' is the π'-approximate viscosity solution, then $\mathrm{v}' = \mathrm{v}$.

□ We also have : comparison/uniqueness + stability

□ Similar results for $D([0, T])$ but with another topology.

□ Compare with Ekren, Touzi, Zhang [7], Ren, Touzi and Zhang [3], and Cosso and Russo [5].

Regularity in the semi-linear case

□ Consider the case where

$$F(t, \mathrm{x}, r, p, q) = f(t, \mathrm{x}, r, p\sigma(t, \mathrm{x})) + b(t, \mathrm{x})p + \frac{1}{2}\sigma^2(t, \mathrm{x})q$$

with

- f, b and σ are Fréchet differentiable with bounded μ_f, μ_b and μ_σ.
- f is C^1_b in (p, q), uniformly.
- g is Fréchet differentiable with bounded derivatives μ_g.
- $(\mathrm{x}, r, p) \mapsto (\mu_g(\cdot; \mathrm{x}), \mu_b(\cdot; t, \mathrm{x}), \mu_\sigma(\cdot; t, \mathrm{x}), \mu_f(\cdot; t, \mathrm{x}, r, p))$ and well as $(\partial_r, \partial_p)f(t, \cdot)$ are uniformly α-Hölder, uniformly in $t \leq T$.

□ **Theorem :** Under the above conditions, $\nabla_{\mathrm{x}}\mathrm{v}$ is well-defined. It is locally uniformly α-Hölder in space (+ uniform continuity in time aparts from atoms of μ_g).

□ In particular, the solution v satisfies

$$\begin{aligned}\mathrm{v}(t,X) =& g(X) + \int_t^T f(s,\mathrm{x},\mathrm{v}(s,X),\nabla_{\mathrm{x}}\mathrm{v}(s,X)\sigma(s,X))ds \\ & - \int_t^T \nabla_{\mathrm{x}}\mathrm{v}(s,X)\sigma(s,X)dW_s, \\ X_t =& X_0 + \int_0^t b(s,X)ds + \int_0^t \sigma(s,X)dW_s.\end{aligned}$$

Thank you !

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