The value of not being predictable

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based on joint work with
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In many optimal control problems arising in Finance, there are moments, sometimes known in advance, when significant new information will become available:

- interest rate decisions by central banks, elections, referendums
- publication of data on GDP growth, job market statistics, . . .
- price jumps, e.g., at opening of exchanges, due to earning announcements, at defaults
- trading algos scanning limit order books for signals of new demand/supply for shares of stock

Reasonable to assume in such moments:

- investors use signals that alert and inform them about impending jumps to take precautionary actions: proactive control;
- afterwards, when the news are fully revealed, further measures may have to be taken: reactive control.
Information flow and optimal control

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- afterwards, when the news are fully revealed, further measures may have to be taken: *reactive control*.

**But**: Standard optimal control only considers *predictable* policies! How to do optimal control with more flexible information flows?
Outline

- Meyer $\sigma$-fields introduced in a toy example: 
  *Risk-neutral optimal investment with position limits*

- Jump signals in a classical control problem: 
  *Merton’s optimal investment problem with jump diffusion*

- Jumps signals in a singular control problem: 
  *Irreversible investment*
Part I

A toy example:
Easier to solve than to formulate
Illustration: Optimal investment with position limits

- asset price fluctuations modeled by symmetric compound Poisson process

\[ P_t = p + \sum_{n=1}^{N_t} Y_n \text{ with i.i.d. } Y_n \sim N_{0,1} \]

- strategy \( \phi = (\phi_t)_{0 \leq t \leq 1} \) with \( |\phi| \leq 1 \) yields expected P&L

\[ \mathbb{E} \int_0^1 \phi_t dP_t = \mathbb{E} \sum_{n=1}^{N_1} \phi_{T_n} Y_n \]

- Question: How to maximize this?
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- **Question:** How to maximize this?
- If controls \( \phi \) are predictable:

\[ \mathbb{E} \int_0^1 \phi_t dP_t \equiv 0 \]

for _any_ control because \( P \) is a martingale
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- Question: How to maximize this?
- If controls \( \phi \) are optional; full info on jumps “as they happen”:
  \[ \mathbb{E} \int_0^1 \phi_t dP_t \leq \mathbb{E} \sum_{n=1}^{N_1} |\phi_{T_n}| |Y_n| \leq \mathbb{E} \sum_{n=1}^{N_1} |Y_n| \]
  with “=” for \( \phi_t^\varnothing = \text{sign}(\Delta P_t) \)
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- Question: How to maximize this?

- If controller alerted only by large enough jumps \( |\Delta P_t| \geq \eta \):

\[ \mathbb{E} \int_0^1 \phi_t dP_t \leq \mathbb{E} \sum_{n=1}^{N_1} |\phi_{T_n}| Y_n 1_{\{Y_{T_n} \geq \eta\}} \leq \mathbb{E} \sum_{n=1}^{N_1} |Y_n| 1_{\{|Y_{T_n} \geq \eta\}} \]

this suggests optimal \( \phi^*_\eta_t = \text{sign}(\Delta P_t 1_{\{|\Delta P_t| \geq \eta\}}) \)
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this suggests optimal \( \phi^\eta_t = \text{sign}(\Delta P_t 1_{\{|\Delta P_t| \geq \eta\}}) \) —but how?
Meyer $\sigma$-fields $\Lambda$

Lenglart ’81: A $\sigma$-field on $\Omega \times [0, \infty)$ is called a Meyer $\sigma$-field if

- it is generated by càdlàg processes $Z$;
- it contains all deterministic Borel-measurable processes;
- it is stable with respect to stopping: with $Z$ also $(Z_{s\wedge t})_{s \geq 0}$ is $\Lambda$-measurable for any $t \geq 0$.

Examples:

$\mathcal{O}$, $\mathcal{P}$,
Meyer $\sigma$-fields $\Lambda$

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$\mathcal{O}$, $\mathcal{P}$, and

$$\Lambda^n = \mathcal{P} \vee \sigma(Z^n) \quad \text{with} \quad Z^n_t := \sum_{n=1}^{N_t} Y_n 1_{\{|Y_n| \geq \eta\}}, \quad t \geq 0$$
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**Examples:**

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**Theorem**

$$\phi^n = \text{sign}(\Delta Z^n) \in \arg \max_{\phi \in \Lambda^n, |\phi| \leq 1} \mathbb{E} \int_0^1 \phi_t \, dP_t$$
Dealing with Meyer $\sigma$-fields

**Proof:**
Observe decomposition of jump times

$$T_n = (T_n)_{|Y_n|\geq \eta} \wedge (T_n)_{|Y_n|<\eta}$$

$\Lambda^\eta$-st.time $\wedge$ $\Lambda^\eta$-tot.inacc.

yields for $\Lambda^\eta$-measurable $\phi$ (with $\phi_{\infty} := 0$):

$$\phi T_n = \phi(T_n)_{|Y_n|\geq \eta} + (\mathcal{P} \phi)(T_n)_{|Y_n|<\eta}$$

So:

$$\mathbb{E} \int_0^1 \phi_t dP_t = \mathbb{E} \sum_{n=1}^{N_1} \phi T_n 1_{|Y_n|\geq \eta} Y_n + \mathbb{E} \sum_{n=1}^{N_1} (\mathcal{P} \phi) T_n 1_{|Y_n|<\eta} Y_n$$

$$\leq \mathbb{E} \sum_{n=1}^{N_1} |\phi T_n| 1_{|Y_{T_n}|\geq \eta} |Y_n|$$

$$= \mathbb{E} \int_0^1 (\mathcal{P} \phi)_t d(P_t - Z_t^\eta) = 0$$

with ‘=’ for $\phi^\eta = \text{sign} \left( \Delta P 1_{|\Delta P| \geq \eta} \right)$
Part II

Merton’s optimal investment problem with jump signals
Merton’s optimal investment problem

Consider an investor who has initial capital $x > 0$ to dynamically invest in

- a savings account bearing interest at rate $r$
- a stock whose price fluctuates according to

$$S_t = S_0 \exp \left( \sigma W_t + \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sum_{n=1}^{N_t} \left( \bar{\sigma} Y_n + \bar{\mu} - \frac{1}{2} \bar{\sigma}^2 \right) \right)$$

for constants $\mu, \sigma > 0$, $W$ a Brownian motion, i.i.d. $Y_n \sim N_{0,1}$ and a Poisson process $N$ with intensity $\lambda > 0$, all independent

...to maximize expected utility from terminal wealth:

$$\mathbb{E}[U(X_T)] \rightarrow \max \text{ with } U(x) = \frac{x^{1-\alpha}}{1-\alpha}, \ x > 0,$$

for some relative risk aversion parameter $\alpha \in (0, \infty) \setminus \{1\}$.
Merton's optimal investment in constant proportions

Well known solution:
It is only admissible to invest a fraction from $[0, 1]$ of one's wealth in the stock at any one time. Among these investment strategies the one investing the constant fraction $\phi^M \in [0, 1]$ maximizing

$$e^M(\phi) := (\mu - r)\phi - \frac{1}{2}\alpha \sigma^2 \phi^2$$

$$+ \lambda \int_\mathbb{R} \left( U(1 + \phi(e^{\bar{\sigma} y + \bar{\mu} - \bar{\sigma}^2} - 1)) - U(1) \right) N_{0,1}(dy)$$

is optimal and it yields the utility

$$v^M(t, x) = U\left(x e^{(r + e^M(\phi^M))(T-t)}\right)$$

when used with initial capital $x > 0$ as of time $t \in [0, T]$. 
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is optimal and it yields the utility

$$V^M(t, x) = U \left( xe^{(r + e^M(\phi_M))(T-t)} \right)$$

when used with initial capital $x > 0$ as of time $t \in [0, T]$.

What changes if we sometimes get a signal on the stock price jump?
Wanted:
Investor has the chance to be alerted by a noisy signal that has a certain correlation with the size of an impending exogenous shock affording the opportunity take a suitable position in the stock before its price jumps.
Accommodating a noisy signal process

**Wanted:**
Investor has the *chance to be alerted* by a noisy signal that has a certain *correlation* with the size of an impending exogenous shock affording the opportunity to take a suitable position in the stock before its price jumps.

**Implementation:**
With independent and i.i.d. \( \delta_n \sim (1 - p)\text{Dirac}_0 + p\text{Dirac}_1 \) for some \( p \in [0, 1] \) and i.i.d. \( \varepsilon_n \sim \mathcal{N}_{0,1} \), use

\[
Z_t := \sum_{n=1}^{N_t} \delta_n (\rho Y_n + \sqrt{1 - \rho^2} \varepsilon_n), \quad t \geq 0,
\]

with \( \rho \in [0, 1] \) to move from predictable investment strategies to strategies measurable with respect to the Meyer \( \sigma \)-field

\[
\Lambda^{\rho,p} = \mathcal{P} \vee \sigma(Z)
\]
Optimal investment with chance of noisy signal

Solution (B.-Körber):

With an unassured and noisy signal, i.e., \( p, \rho \in [0, 1) \), it is only admissible to invest a fraction from \([0, 1]\) of one’s wealth in the stock at any one time. Among these investment strategies the one that invests the constant fraction \( \phi^0 \in [0, 1] \) maximizing

\[
e^0(\phi) := (\mu - r)\phi - \frac{1}{2} \alpha \sigma^2 \phi^2
+ (1 - p)\lambda \int_{\mathbb{R}} \left( U(1 + \phi(e^{\bar{y}+\bar{\mu}-\bar{\sigma}^2/2} - 1)) - U(1) \right) N_{0,1}(dy)
\]

at times without jump signal and that, at moments when given a signal \( z = \Delta \bar{Z}_t \neq 0 \), invests the fraction \( \phi^z \) maximizing

\[
e^z(\phi) := \int_{\mathbb{R}} \left( U(1 + \phi(e^{\bar{y}+\bar{\mu}-\bar{\sigma}^2/2} - 1)) - U(1) \right) N_{\rho^z,1-\rho^2}(dy)
\]

is optimal.
Optimal investment with chance of noisy signal

- Merton investment level without signal: $p = 0$
- Investor becomes more aggressive the higher the alert likelihood $p$
- But exceeding position limit of 100% $\mu \sigma^2$ inadmissible
- Investment nondecreasing with respect to signal
- Signal useful even when $\rho = 0$: jump alert
- Disinvestment even for positive signal: hedging against jumps
Optimal investment with chance of noisy signal

Solution (B.-Körber) (ctd):
This policy yields the indirect utility

\[ v(t, x) = U \left( xe^{r+e^0(\phi^0)+\rho \lambda \int_{\mathbb{R}} e^z(\phi^z)N_{0,1}(dz)}(T-t) \right) \]

when used as of time \( t \in [0, T] \).

- \( e(0, \rho) = e^M(\phi^M) \): Merton investor’s indifference mark up due to stock investment possibility
- \( e(p, 0) > 0 \): jump alerts afford hedging opportunity
- improvements of alert likelihood \( p \) similarly effective as improvements of signal quality \( \rho \)
Optimal investment with chance of noisy signal

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The value of not being predictable.
Part III

Exogenous vs. endogenous jumps
Proactive and reactive controls causing jumps

**New aspect:** Timing of shocks and interventions

**So far:** Controlled system only exhibits exogenous jumps and the control just determines our exposure to the ensuing shocks

∼ classical control problem: Merton problem

**But:** What if in addition our controls can cause an endogenous jump in the system?

∼ singular control problem: Merton with proportional transaction costs

**Issue:** What if such an endogenous jump coincides with an exogenous one? Is the control jump a reaction to an exogenous shock—or is it a proactive intervention in preparation to such a shock?
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**Answer:** It can be both—
Proactive and reactive controls causing jumps

New aspect: **Timing** of shocks and interventions

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**Issue:** What if such an endogenous jump coincides with an exogenous one? Is the control jump a reaction to an exogenous shock—or is it a proactive intervention in preparation to such a shock?

**Answer:** It can be both—and the proactive control jump can even take into account extra information about the exogenous jump when we use Meyer $\sigma$-fields!
Singular control problem: Irreversible investment

- Classic problem: Dixit and Pindyck ('94), Bertola (1998), Merhi and Zervos ('07), Riedel and Su ('11), Ferrari ('15), Al Motairi and Zervos ('17), De Angelis et al. ('17) . . .

- Consider target functional:

\[
\tilde{V}(C) = \mathbb{E} \left[ \int_{[0,\infty)} \tilde{P}_t \, dC_t - \int_{[0,\infty)} \rho_t(C_t) \, dR_t \right] \to \max_{C \geq c_0}
\]

\[\tilde{P}\] discounted reward process, \(\rho_t(c)\) risk penalty convex in \(c\), \(R\) risk assessment clock

- **Standard** assumptions: \(\tilde{P}_t = e^{-rt}P_t\) for compound Poisson \(P_t = p + \sum_{k=1}^{N_t} Y_k\); \(\rho_t(c) = c^2/2\); \(dR_t = e^{-rt}dt\):

\[
\mathbb{E} \left[ \int_{[0,\infty)} e^{-rt} P_t \, dC_t - \int_{[0,\infty)} \frac{1}{2}(C_t)^2 e^{-rt} \, dt \right] \to \max_{C \geq c_0} \text{in } \mathcal{P}
\]
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\( \tilde{P} \) discounted reward process, \( \rho_t(c) \) risk penalty convex in \( c \), \( R \) risk assessment clock

- Alternative assumptions: \( \tilde{P}_t = e^{-rt} P_t \) for compound Poisson \( P_t = p + \sum_{k=1}^{N_t} Y_k; \rho_t(c) = c^2/2; dR_t = e^{-rt} dN_t: \)

\[ \mathbb{E} \left[ \int_{[0,\infty)} e^{-rt} P_t \, dC_t - \int_{[0,\infty)} \frac{1}{2} (C_t)^2 e^{-rt} \, dN_t \right] \rightarrow \max_{C \geq c_0} \text{ in } \Lambda \]

- Proactive and reactive control: \( C \) làdlàg required!

Signal on large jumps: \( \Lambda^\eta = \mathcal{P} \lor \sigma \left( \sum_{n=1}^{N_1} Y_n 1_{\{|Y_n| \geq \eta\}} \right) \).
Heuristics from first order conditions

First order conditions for optimality of $\hat{C}$:

$$E \left[ \tilde{P}_S \middle| \mathcal{F}_S^\Lambda \right] \leq E \left[ \int_{[S,\infty)} \frac{\partial}{\partial c} \rho_t(\hat{C}_t) dR_t \middle| \mathcal{F}_S^\Lambda \right]$$

with “=” holding true whenever it is optimal to intervene: $d\hat{C}_S > 0$
Heuristics from first order conditions

First order conditions for optimality of $\hat{C}$:

$$
\mathbb{E} \left[ \tilde{P}_S \mid \mathcal{F}^\Lambda_S \right] \leq \mathbb{E} \left[ \int_{[S,\infty)} \frac{\partial}{\partial c} \rho_t(\hat{C}_t) dR_t \bigg| \mathcal{F}^\Lambda_S \right]
$$

with "=" holding true whenever it is optimal to intervene: $d\hat{C}_S > 0$

If optimal to intervene at $S$, then for any $T$ with $T > S$:

$$
\mathbb{E} \left[ \tilde{P}_S - \tilde{P}_T \mid \mathcal{F}^\Lambda_S \right] \geq \mathbb{E} \left[ \int_{[S,T)} \frac{\partial}{\partial c} \rho_t(\hat{C}_t) dR_t \bigg| \mathcal{F}^\Lambda_S \right] \geq \mathbb{E} \left[ \int_{[S,T)} \frac{\partial}{\partial c} \rho_t(\hat{C}_S) dR_t \bigg| \mathcal{F}^\Lambda_S \right]
$$

$\sim \hat{C}_S \leq \text{ess inf}_T \ell^\Lambda_{S,T} =: L^\Lambda_S$
Heuristics from first order conditions

First order conditions for optimality of $\hat{C}$:

$$
\mathbb{E} \left[ \tilde{P}_S \middle| \mathcal{F}_S^\Lambda \right] \leq \mathbb{E} \left[ \int_{[S, \infty)} \frac{\partial}{\partial c} \rho_t(\hat{C}_t) dR_t \middle| \mathcal{F}_S^\Lambda \right]
$$

with "=" holding true whenever it is optimal to intervene: $d\hat{C}_S > 0$

If not optimal to intervene at $S$, then for next optimal intervention time $T_S$ we get

$$
\mathbb{E} \left[ \tilde{P}_S - \tilde{P}_{T_S} \middle| \mathcal{F}_S^\Lambda \right] \leq \mathbb{E} \left[ \int_{[S, T_S)} \frac{\partial}{\partial c} \rho_t(\hat{C}_t) dR_t \middle| \mathcal{F}_S^\Lambda \right] = \mathbb{E} \left[ \int_{[S, T_S)} \frac{\partial}{\partial c} \rho_t(\hat{C}_S) dR_t \middle| \mathcal{F}_S^\Lambda \right]
$$

$\leadsto \hat{C}_S \geq \ell^\Lambda_{S, T_S} \geq \text{ess inf}_T \ell^\Lambda_{S, T} =: L^\Lambda_S$

In conjunction with ‘≤’ from before: $\hat{C}_S = c_0 \vee \sup_{v \in [0, S]} L^\Lambda_v$
A stochastic representation theorem

Theorem (B.-El Karoui (2004), B.-Besslich (2021+))

Under suitable integrability and upper-semicontinuity assumptions, there exists $L^\Lambda \in \Lambda$ such that

$$
\mathbb{E} \left[ \tilde{P}_S \bigg| \mathcal{F}^\Lambda_S \right] = \mathbb{E} \left[ \int_{[S,\infty)} \frac{\partial}{\partial c} \rho_t \left( \sup_{v \in [S,t]} L^\Lambda_v \right) dR_t \bigg| \mathcal{F}^\Lambda_S \right], \quad S \in \mathcal{I}^\Lambda.
$$

The maximal such $L^\Lambda$ is uniquely determined by

$$
L^\Lambda_S = \text{essinf}_{T \in \mathcal{I}^\Lambda, T > S} \ell^\Lambda_{S,T}, \quad S \in \mathcal{I}^\Lambda,
$$

where for $S < T$, $\ell^\Lambda_{S,T} \in \mathcal{F}^\Lambda_S$ is defined by

$$
\mathbb{E} \left[ \tilde{P}_S - \tilde{P}_T \bigg| \mathcal{F}^\Lambda_S \right] = \mathbb{E} \left[ \int_{[S,T)} \frac{\partial}{\partial c} \rho_t(\ell^\Lambda_{S,T}) dR_t \bigg| \mathcal{F}^\Lambda_S \right]
$$

on $\{\mathbb{P} \left[ R_{T-} - R_{S-} > 0 \big| \mathcal{F}^\Lambda_S \right] > 0 \}$ and $\ell^\Lambda_{S,T} := \infty$ elsewhere.
Explicit solution in the compound Poisson example

Let \( \tilde{P}_t = e^{-rt} P_t \) with \( P_t = p + \sum_{k=1}^{N_t} Y_k \) for Poisson \( N \) with param. \( \lambda \), i.i.d. \( Y_k \in L^2, \mathbb{E} Y_k = m; dR_t = e^{-rt} dN_t; \rho_t(c) = \frac{1}{2} c^2; \)

\[
\Lambda = \Lambda^\eta := \mathcal{P} \vee \sigma \left( \sum_{k=1}^{N_t} Y_k 1_{\{|Y_k| \geq \eta\}} \right) \sim \text{Large jump alerts}
\]

Probability of alert: \( p(\eta) = \mathbb{P}[|Y_k| \geq \eta]. \)

- \( p(\eta) = 0: \) no alerts, predictable case \( \Lambda^\eta = \mathcal{P} \)
- \( p(\eta) = 1: \) alerts for all jumps, optional case \( \Lambda^\eta = \mathcal{O} \)
- \( p(\eta) \in (0, 1): \) Meyer case \( \mathcal{P} \subsetneq \Lambda^\eta \subsetneq \mathcal{O} \)
Solution in the predictable case

In the case $p(\eta) = 0$, i.e. without alerts:

$$L_t^\mathcal{P} = a(P_{t^-} - b), \quad t \in [0, \infty),$$

where the constants $a, b$ are given by

$$a := \frac{1}{\mathbb{E}[R_{\infty^-}]} = \frac{r}{\lambda},$$

$$b := \sup_{0 < T \text{ pred.}} \left[ \frac{\mathbb{E} \left[ e^{-rT} \sum_{k=1}^{N_T} Y_k \right]}{1 - \mathbb{E} \left[ e^{-rT} \right]} \right] = \frac{\mathbb{E} \left[ \int_{[0, \infty)} \left( \sup_{v \in [0, t]} P_v - p \right) \, dR_t \right]}{\mathbb{E}[R_{\infty^-}]}.$$ 

$$\leadsto C^\mathcal{P} = c_0 \vee \sup_{0 \leq v \leq \cdot} L_v^\mathcal{P} \quad \text{left-continuous with exclusively reactive jumps because jump times are totally inaccessible to controller}$$
Solution in the Meyer case

In the case $p(\eta) \in (0, 1)$ with alerts for some, but not all jumps:

$$L^\eta_t = \begin{cases} 
0, & P^\eta_t \geq b, \ |\Delta P^\eta_t| \geq \eta, \\
\frac{\xi}{\lambda}(P^\eta_t - b), & P^\eta_t \geq b, \ |\Delta P^\eta_t| < \eta, \\
\arg\inf_{\gamma^0 \in (0, B_0^\eta \cdot (b - P^\eta_t))} f_1^\eta(\gamma^0, 0, P^\eta_t) < 0, & P^\eta_t < b, \ |\Delta P^\eta_t| \geq \eta, \\
\arg\inf_{\gamma^1 \in (-B_1^\eta \cdot (b - P^\eta_t), 0)} f_0^\eta(0, \gamma^1, P^\eta_t) < 0, & P^\eta_t < b, \ |\Delta P^\eta_t| < \eta
\end{cases}$$

where $P^\eta_t := P_{t^-} + \Delta P_t 1_{\{\Delta P_t \geq \eta\}}$, $t \geq 0$,

$$f^\eta_\Delta = \frac{\left(1 - \mathbb{E} \left[ e^{-r T^\eta(\gamma^0, \gamma^1)} \right] \right) P - \mathbb{E} \left[ e^{-r T^\eta(\gamma^0, \gamma^1)} \sum_{k=1}^{N_{T^\eta(\gamma^0, \gamma^1)}} Y_k \right]}{\frac{\lambda}{r} \left(1 - \mathbb{E} \left[ e^{-r T^\eta(\gamma^0, \gamma^1)} \right] \right) - \mathbb{E} \left[ e^{-r T^\eta(\gamma^0, \gamma^1)} 1_{\{\Delta P_{T^\eta(\gamma^0, \gamma^1)} \geq \eta\}} \right] + \Delta},$$

$$T^\eta(\gamma^0, \gamma^1) = \inf \left\{ t \in \{\Lambda^\eta N > 0\} \mid (|\Delta P_t| < \eta \text{ and } P_{t^-} - p \geq \gamma^0) \right\} \text{ or } (|\Delta P_t| \geq \eta \text{ and } P_t - p \geq \gamma^1).$$
Solution in the optional case

In the optional case with alerts for all jumps $p(\eta) = 1$:

$$L_t^\theta = \begin{cases} 
0, & P_t \geq b, \ |\Delta P_t| > 0, \\
\frac{r}{\lambda}(P_t - b), & P_t \geq b, \Delta P_t = 0, \\
\frac{r}{\lambda + r}(b - P_t), & P_t < b, \ |\Delta P_t| > 0, \\
\inf_{\gamma \in (-\infty,0)} f(\gamma, P_t) < 0, & m\frac{\lambda}{r} \leq P_t < b, \Delta P_t = 0, \\
-\infty, & P_t < m\frac{\lambda}{r}, \Delta P_t = 0.
\end{cases}$$

where

$$f(\gamma, P_t) := \frac{(1 - \mathbb{E}[e^{-rT(\gamma)}]) \ p - \mathbb{E} \left[ e^{-rT(\gamma)} \sum_{k=1}^{N_{T(\gamma)}} Y_k \right]}{\frac{\lambda}{r} \left(1 - \mathbb{E}[e^{-rT(\gamma)}]\right) - \mathbb{E}[e^{-rT(\gamma)}]}.$$ 

$$T(\gamma) := \inf \left\{ t \in \{N > 0\} \middle| |\Delta P_t| > 0 \text{ and } P_t - p \geq \gamma \right\}.$$
Solution in the optional case

In the optional case with alerts for all jumps \( p(\eta) = 1 \):

\[
L_t^\phi = \begin{cases}
0, & P_t \geq b, \ |\Delta P_t| > 0, \\
\frac{r}{\lambda}(P_t - b), & P_t \geq b, \ \Delta P_t = 0, \\
\frac{r}{\lambda + r}(b - P_t), & P_t < b, \ |\Delta P_t| > 0, \\
\inf_{\gamma \in (-\infty, 0)} f(\gamma, P_t) < 0, & m_\lambda \leq P_t < b, \ \Delta P_t = 0, \\
-\infty, & P_t < m_\lambda \frac{\lambda}{r}, \ \Delta P_t = 0.
\end{cases}
\]

where

\[
f(\gamma, P_t) := \frac{(1 - \mathbb{E} [e^{-rT(\gamma)}]) p - \mathbb{E} \left[ e^{-rT(\gamma)} \sum_{k=1}^{N_{T(\gamma)}} Y_k \right]}{\frac{\lambda}{r} (1 - \mathbb{E} [e^{-rT(\gamma)}]) - \mathbb{E} [e^{-rT(\gamma)}]},
\]

\[
T(\gamma) := \inf \left\{ t \in \{N > 0\} \mid |\Delta P_t| > 0 \text{ and } P_t - p \geq \gamma \right\}.
\]

Observation: \( L_t^\phi \xleftarrow{p(\eta)\rightarrow 1} L_{\hat{\eta}} \xrightarrow{p(\eta)\rightarrow 0} L_t^\phi \)
Figure: $P^\eta$ (black), $b$ (Magenta) and optimal controls for $\eta = 0$ (blue, optional), $\eta = 3$, $\eta = 6$ (green) and $\eta = \infty$ (red, predictable). The dots indicate the processes’ value at their jump times.
Figure: The value of not being predictable.
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- with **Körber**: Merton’s optimal investment problem with jump signals (working title), in preparation
Conclusion and Outlook

- continuous-time information modeling most flexibly done via Meyer $\sigma$-fields
- allows for modeling signals on jump events and for strategies to act on them: interpolation between predictable and optional information flow
- Merton’s optimal investment problem with signals on jumps: there is value in not being predictable in classical control problems
- Irreversible investment: explicit solution in compound Poisson setting with jump size dependent alerts; pro-active and re-active interventions in a singular control with jump signals

Thank you very much!
Conclusion and Outlook

- continuous-time information modeling most flexibly done via Meyer $\sigma$-fields
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- Irreversible investment: explicit solution in compound Poisson setting with jump size dependent alerts; pro-active and re-active interventions in a singular control with jump signals
- ongoing: duality in model with transient price impact, more tools for dynamic programming, further applications

Thank you very much!
Optimal stopping

\[ \mathbb{E} \left[ X_T + \int_{[0,T)} g_t(\ell) \mu(dt) \right] \to \max_{T \in \mathcal{F}^\Lambda} \]

- Reward upon stopping: \( \Lambda \)-measurable \( X \) of class \( D^\Lambda \)
- Running reward: \( g = g_t(\omega, \ell) \): strictly & continuously increasing from \(-\infty\) to \(+\infty\) in parameter \( \ell \in (-\infty, +\infty) \)
- Payment clock: \( \mu(dt) \), possibly random and with atoms with \( \mathbb{E}[\int_{[0,\infty)} g_t(\ell) \mu(dt)] < \infty \) for any \( \ell \)

Questions: Existence and construction of optimal stopping time?
Note: Inevitable lack of upper-semi-continuity if \( \mu \) has atoms!
Optimal stopping

\[ \mathbb{E} \left[ X_\tau + \int_{[0,\tau)} g_t(\ell) \mu(dt) \right] \to \max_{\tau \in \mathcal{S}^\Lambda, \text{div}} \]

- Reward upon stopping: \( \Lambda \)-measurable \( X \) of class \( D^\Lambda \)
- Running reward: \( g = g_t(\omega, \ell) \): strictly & continuously increasing from \(-\infty\) to \(+\infty\) in parameter \( \ell \in (-\infty, +\infty) \)
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Questions: Existence and construction of optimal stopping time?
Note: Inevitable lack of upper-semi-continuity if \( \mu \) has atoms!
El Karoui’s relaxation to divided stopping times considers

\[ \tau = (T, H^l, H, H^r) \text{ with } X_\tau = \ast X_T 1_{H^l} + X_T 1_H + X_T^* 1_{H^r} \]
Theorem

Let a “suitably u.s.c.” \( X \) be represented by \( L \) as

\[
X_S = \mathbb{E} \left[ \int_{[S, \infty)} g_t(\sup_{v \in [S,t]} L_v)\mu(dt) \middle| \mathcal{F}_S^\Lambda \right], \quad S \in \mathcal{I}^\Lambda.
\]

Then both

\[
\tau^{\geq \ell} = (T^{\geq \ell}, \emptyset, H^{\geq \ell}, \Omega \backslash (H^{\geq \ell})) \quad \text{and} \quad \tau^{> \ell} = (T^{> \ell}, \emptyset, H^{> \ell}, \Omega \backslash (H^{> \ell}))
\]

given, respectively, by

\[
T^{\geq \ell} = \inf \{ t \geq 0 \mid \sup_{[0,t]} L \geq \ell \}, \quad H^{\geq \ell} = \{ L_{T^{\geq \ell}} \geq \ell \},
\]

\[
T^{> \ell} = \inf \{ t \geq 0 \mid \sup_{[0,t]} L > \ell \}, \quad H^{> \ell} = \{ L_{T^{> \ell}} > \ell \},
\]

are optimal for the stopping problem with parameter \( \ell \).
Illustration

Figure: $\tilde{P}^\eta$ (black), $b$ (Magenta) and optimal controls for $\eta = 0$ (blue, optional), $\eta = 3$, $\eta = 6$ (green) and $\eta = \infty$ (red, predictable). The dots indicate the processes' value at their jump times.