# The value of not being predictable

Peter Bank

Technische Universität Berlin

based on joint work with

David Besslich and Laura Körber

Bachelier Finance Society One World Seminar

Cyberspace, 20 May 2021

# Information flow and optimal control

In many optimal control problems arising in Finance, there are moments, sometimes known in advance, when significant new information will become available:

- interest rate decisions by central banks, elections, referendums
- publication of data on GDP growth, job market statistics, ...
- price jumps, e.g., at opening of exchanges, due to earning announcements, at defaults
- trading algos scanning limit order books for signals of new demand/supply for shares of stock

Reasonable to assume in such moments:

- investors use signals that alert and inform them about impending jumps to take precautionary actions: *proactive control*;
- afterwards, when the news are fully revealed, further measures may have to be taken: *reactive control*.

# Information flow and optimal control

In many optimal control problems arising in Finance, there are moments, sometimes known in advance, when significant new information will become available:

- interest rate decisions by central banks, elections, referendums
- publication of data on GDP growth, job market statistics, ...
- price jumps, e.g., at opening of exchanges, due to earning announcements, at defaults
- trading algos scanning limit order books for signals of new demand/supply for shares of stock

Reasonable to assume in such moments:

- investors use signals that alert and inform them about impending jumps to take precautionary actions: *proactive control*;
- afterwards, when the news are fully revealed, further measures may have to be taken: *reactive control*.

But: Standard optimal control only considers predictable policies!

How to do optimal control with more flexible information flows?

# Outline

- Meyer $\sigma$-fields introduced in a toy example:
  *Risk-neutral optimal investment with position limits*
- Jump signals in a classical control problem:
  *Merton's optimal investment problem with jump diffusion*
- Jumps signals in a singular control problem:
  *Irreversible investment*

# Part I

## A toy example:
## Easier to solve than to formulate

# Illustration: Optimal investment with position limits

- asset price fluctuations modeled by symmetric compound Poisson process

$$P_t = p + \sum_{n=1}^{N_t} Y_n \text{ with i.i.d. } Y_n \sim N_{0,1}$$

- strategy $\phi = (\phi_t)_{0 \leq t \leq 1}$ with $|\phi| \leq 1$ yields expected P&L

$$\mathbb{E} \int_0^1 \phi_t dP_t = \mathbb{E} \sum_{n=1}^{N_1} \phi_{T_n} Y_n$$

- Question: How to maximize this?

# Illustration: Optimal investment with position limits

- asset price fluctuations modeled by symmetric compound Poisson process

$$P_t = p + \sum_{n=1}^{N_t} Y_n \text{ with i.i.d. } Y_n \sim N_{0,1}$$

- strategy $\phi = (\phi_t)_{0 \leq t \leq 1}$ with $|\phi| \leq 1$ yields expected P&L

$$\mathbb{E} \int_0^1 \phi_t dP_t = \mathbb{E} \sum_{n=1}^{N_1} \phi_{T_n} Y_n$$

- Question: How to maximize this?
- If controls $\phi$ are predictable:

$$\mathbb{E} \int_0^1 \phi_t dP_t \equiv 0$$

for *any* control because $P$ is a martingale

# Illustration: Optimal investment with position limits

- asset price fluctuations modeled by symmetric compound Poisson process

$$P_t = p + \sum_{n=1}^{N_t} Y_n \text{ with i.i.d. } Y_n \sim N_{0,1}$$

- strategy $\phi = (\phi_t)_{0 \leq t \leq 1}$ with $|\phi| \leq 1$ yields expected P&L

$$\mathbb{E} \int_0^1 \phi_t dP_t = \mathbb{E} \sum_{n=1}^{N_1} \phi_{T_n} Y_n$$

- Question: How to maximize this?
- If controls $\phi$ are optional; full info on jumps "as they happen":

$$\mathbb{E} \int_0^1 \phi_t dP_t \leq \mathbb{E} \sum_{n=1}^{N_1} |\phi_{T_n}||Y_n| \leq \mathbb{E} \sum_{n=1}^{N_1} |Y_n|$$

with "$=$" for $\phi_t^{\mathscr{O}} = \mathrm{sign}(\Delta P_t)$

# Illustration: Optimal investment with position limits

- asset price fluctuations modeled by symmetric compound Poisson process

$$P_t = p + \sum_{n=1}^{N_t} Y_n \text{ with i.i.d. } Y_n \sim N_{0,1}$$

- strategy $\phi = (\phi_t)_{0 \leq t \leq 1}$ with $|\phi| \leq 1$ yields expected P&L

$$\mathbb{E} \int_0^1 \phi_t dP_t = \mathbb{E} \sum_{n=1}^{N_1} \phi_{T_n} Y_n$$

- Question: How to maximize this?
- If controller alerted only by large enough jumps $|\Delta P_t| \geq \eta$:

$$\mathbb{E} \int_0^1 \phi_t dP_t \overset{?!}{\leq} \mathbb{E} \sum_{n=1}^{N_1} |\phi_{T_n}||Y_n| 1_{\{|Y_{T_n} \geq \eta\}} \leq \mathbb{E} \sum_{n=1}^{N_1} |Y_n| 1_{\{|Y_{T_n} \geq \eta\}}$$

this suggests optimal $\phi_t^\eta = \mathrm{sign}(\Delta P_t 1_{\{|\Delta P_t| \geq \eta\}})$

# Illustration: Optimal investment with position limits

- asset price fluctuations modeled by symmetric compound Poisson process

$$P_t = p + \sum_{n=1}^{N_t} Y_n \text{ with i.i.d. } Y_n \sim N_{0,1}$$

- strategy $\phi = (\phi_t)_{0 \leq t \leq 1}$ with $|\phi| \leq 1$ yields expected P&L

$$\mathbb{E} \int_0^1 \phi_t dP_t = \mathbb{E} \sum_{n=1}^{N_1} \phi_{T_n} Y_n$$

- Question: How to maximize this?
- If controller alerted only by large enough jumps $|\Delta P_t| \geq \eta$:

$$\mathbb{E} \int_0^1 \phi_t dP_t \overset{?!}{\leq} \mathbb{E} \sum_{n=1}^{N_1} |\phi_{T_n}||Y_n| 1_{\{|Y_{T_n} \geq \eta\}} \leq \mathbb{E} \sum_{n=1}^{N_1} |Y_n| 1_{\{|Y_{T_n} \geq \eta\}}$$

this suggests optimal $\phi_t^\eta = \mathrm{sign}(\Delta P_t 1_{\{|\Delta P_t| \geq \eta\}})$ —but how?

# Meyer $\sigma$-fields $\Lambda$

Lenglart '81: A $\sigma$-field on $\Omega \times [0, \infty)$ is called a **Meyer $\sigma$-field** if

- it is generated by càdlàg processes $Z$;
- it contains all deterministic Borel-measurable processes;
- it is stable with respect to stopping: with $Z$ also $(Z_{s \wedge t})_{s \geq 0}$ is $\Lambda$-measurable for any $t \geq 0$.

Examples:

$\mathscr{O}$, $\mathscr{P}$,

# Meyer $\sigma$-fields $\Lambda$

Lenglart '81: A $\sigma$-field on $\Omega \times [0, \infty)$ is called a **Meyer $\sigma$-field** if

- it is generated by càdlàg processes $Z$;
- it contains all deterministic Borel-measurable processes;
- it is stable with respect to stopping: with $Z$ also $(Z_{s \wedge t})_{s \geq 0}$ is $\Lambda$-measurable for any $t \geq 0$.

Examples:

$\mathscr{O}$, $\mathscr{P}$, and

$$\Lambda^\eta = \mathscr{P} \vee \sigma\left(Z^\eta\right) \text{ with } Z_t^\eta := \sum_{n=1}^{N_t} Y_n 1_{\{|Y_n| \geq \eta\}}, \ t \geq 0$$

# Meyer $\sigma$-fields $\Lambda$

Lenglart '81: A $\sigma$-field on $\Omega \times [0, \infty)$ is called a **Meyer $\sigma$-field** if

- it is generated by càdlàg processes $Z$;
- it contains all deterministic Borel-measurable processes;
- it is stable with respect to stopping: with $Z$ also $(Z_{s \wedge t})_{s \geq 0}$ is $\Lambda$-measurable for any $t \geq 0$.

Examples:

$\mathscr{O}$, $\mathscr{P}$, and

$$\Lambda^\eta = \mathscr{P} \vee \sigma\left(Z^\eta\right) \text{ with } Z_t^\eta := \sum_{n=1}^{N_t} Y_n 1_{\{|Y_n| \geq \eta\}}, \; t \geq 0$$

Theorem

$$\phi^\eta = \operatorname{sign}\left(\Delta Z^\eta\right) \in \underset{\phi \in \Lambda^\eta, \, |\phi| \leq 1}{\arg\max} \, \mathbb{E} \int_0^1 \phi_t dP_t$$

# Dealing with Meyer $\sigma$-fields

Proof:

Observe decomposition of jump times

$$T_n = \underbrace{(T_n)_{\{|Y_n|\geq\eta\}}}_{\Lambda^\eta\text{-st.time}} \wedge \underbrace{(T_n)_{\{|Y_n|<\eta\}}}_{\Lambda^\eta\text{-tot.inacc.}}$$

yields for $\Lambda^\eta$-measurable $\phi$ (with $\phi_\infty := 0$):

$$\phi_{T_n} = \phi_{(T_n)_{\{|Y_n|\geq\eta\}}} + ({}^{\mathscr{P}}\phi)_{(T_n)_{\{|Y_n|<\eta\}}}$$

So:

$$\mathbb{E}\int_0^1 \phi_t dP_t = \underbrace{\mathbb{E}\sum_{n=1}^{N_1} \phi_{T_n} 1_{\{|Y_n|\geq\eta\}} Y_n}_{\leq \mathbb{E}\sum_{n=1}^{N_1} |\phi_{T_n}| 1_{\{|Y_{T_n}|\geq\eta\}} |Y_n|} + \underbrace{\mathbb{E}\sum_{n=1}^{N_1} ({}^{\mathscr{P}}\phi)_{T_n} 1_{\{|Y_n|<\eta\}} Y_n}_{=\mathbb{E}\int_0^1 ({}^{\mathscr{P}}\phi)_t d(P_t - Z_t^\eta)=0}$$

with '$=$' for $\phi^\eta = \operatorname{sign}\left(\Delta P 1_{\{|\Delta P|\geq\eta\}}\right)$ □

# Part II

# Merton's optimal investment problem with jump signals

# Merton's optimal investment problem

Consider an investor who has initial capital $x > 0$ to dynamically invest in

- a savings account bearing interest at rate $r$
- a stock whose price fluctuates according to

  $$S_t = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sum_{n=1}^{N_t} \left(\bar{\sigma} Y_n + \bar{\mu} - \frac{1}{2}\bar{\sigma}^2\right)\right)$$

  for constants $\mu$, $\sigma > 0$, $W$ a Brownian motion, i.i.d. $Y_n \sim N_{0,1}$ and a Poisson process $N$ with intensity $\lambda > 0$, all independent

to maximize expected utility from terminal wealth:

$$\mathbb{E}[U(X_T)] \to \max \text{ with } U(x) = \frac{x^{1-\alpha}}{1-\alpha},\ x > 0,$$

for some relative risk aversion parameter $\alpha \in (0, \infty)\backslash\{1\}$.

# Merton's optimal investment in constant proportions

Well known solution:

It is only admissible to invest a fraction from $[0,1]$ of one's wealth in the stock at any one time. Among these investment strategies the one investing the constant fraction $\phi^M \in [0,1]$ maximizing

$$
\begin{aligned}
e^M(\phi) :=& (\mu - r)\phi - \frac{1}{2}\alpha\sigma^2\phi^2 \\
&+ \lambda \int_{\mathbb{R}} \left( U(1 + \phi(e^{\bar{\sigma}y + \bar{\mu} - \frac{\bar{\sigma}^2}{2}} - 1)) - U(1) \right) N_{0,1}(dy)
\end{aligned}
$$

is optimal and it yields the utility

$$
v^M(t,x) = U\left( x e^{(r + e^M(\phi^M))(T-t)} \right)
$$

when used with initial capital $x > 0$ as of time $t \in [0,T]$.

# Merton's optimal investment in constant proportions

Well known solution:

It is only admissible to invest a fraction from $[0,1]$ of one's wealth in the stock at any one time. Among these investment strategies the one investing the constant fraction $\phi^M \in [0,1]$ maximizing

$$e^M(\phi) := (\mu - r)\phi - \frac{1}{2}\alpha\sigma^2\phi^2 + \lambda \int_{\mathbb{R}} \left( U(1 + \phi(e^{\bar{\sigma}y + \bar{\mu} - \frac{\bar{\sigma}^2}{2}} - 1)) - U(1) \right) N_{0,1}(dy)$$

is optimal and it yields the utility

$$v^M(t,x) = U\left(xe^{(r + e^M(\phi^M))(T-t)}\right)$$

when used with initial capital $x > 0$ as of time $t \in [0, T]$.

What changes if we sometimes get a signal on the stock price jump?

# Accommodating a noisy signal process

Wanted:

Investor has the *chance to be alerted* by a noisy signal that has a certain *correlation* with the size of an impending exogenous shock affording the opportunity take a suitable position in the stock before its price jumps

# Accommodating a noisy signal process

Wanted:

Investor has the *chance to be alerted* by a noisy signal that has a certain *correlation* with the size of an impending exogenous shock affording the opportunity take a suitable position in the stock before its price jumps

Implementation:

With independent and i.i.d. $\delta_n \sim (1-p)\mathrm{Dirac}_0 + p\mathrm{Dirac}_1$ for some $p \in [0,1]$ and i.i.d. $\varepsilon_n \sim N_{0,1}$, use

$$Z_t := \sum_{n=1}^{N_t} \delta_n(\rho Y_n + \sqrt{1-\rho^2}\varepsilon_n), \quad t \geq 0,$$

with $\rho \in [0,1]$ to move from predictable investment strategies to strategies measurable with respect to the Meyer $\sigma$-field

$$\Lambda^{\rho,p} = \mathscr{P} \vee \sigma(Z)$$

# Optimal investment with chance of noisy signal

## Solution (B.-Körber):

With an unassured and noisy signal, i.e., $p, \rho \in [0,1)$, it is only admissible to invest a fraction from $[0,1]$ of one's wealth in the stock at any one time. Among these investment strategies the one that invests the constant fraction $\phi^0 \in [0,1]$ maximizing

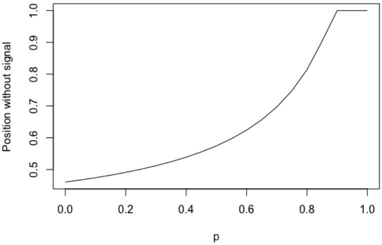
$$
\begin{aligned}
e^0(\phi) &:= (\mu - r)\phi - \frac{1}{2}\alpha\sigma^2\phi^2 \\
&\quad + (1-p)\lambda \int_{\mathbb{R}} \Big( U(1 + \phi(e^{\bar{\sigma} y + \bar{\mu} - \bar{\sigma}^2/2} - 1)) - U(1) \Big) N_{0,1}(dy)
\end{aligned}
$$

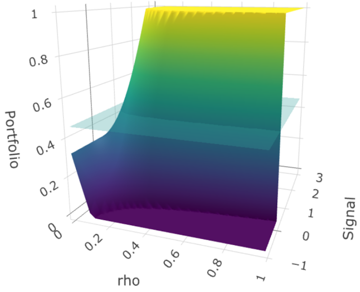
at times without jump signal and that, at moments when given a signal $z = \Delta Z_t \neq 0$, invests the fraction $\phi^z$ maximizing

$$
e^z(\phi) := \int_{\mathbb{R}} \Big( U(1 + \phi(e^{\bar{\sigma} y + \bar{\mu} - \bar{\sigma}^2/2} - 1)) - U(1) \Big) N_{\rho z, 1-\rho^2}(dy)
$$

is optimal.

# Optimal investment with chance of noisy signal




- Merton investment level without signal: $p = 0$
- investor becomes more aggressive the higher the alert likelihood $p$
- but exceeding position limit of 100% $< \frac{\mu}{\alpha\sigma^2}$ inadmissible
- investment nondecreasing with respect to signal
- signal useful even when $\rho = 0$: jump alert
- disinvestment even for positive signal: hedging against jumps

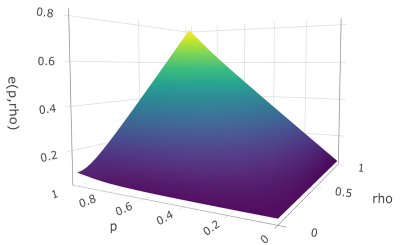
# Optimal investment with chance of noisy signal

## Solution (B.-Körber) (ctd):

This policy yields the indirect utility

$$v(t,x) = U\left(xe^{\left(r+e^0(\phi^0)+p\lambda\int_{\mathbb{R}} e^z(\phi^z)N_{0,1}(dz)\right)(T-t)}\right)$$

when used as of time $t \in [0, T]$.




- $e(0,\rho) = e^M(\phi^M)$: Merton investor's indifference mark up due to stock investment possibility
- $e(p,0) > 0$: jump alerts afford hedging opportunity
- improvements of alert likelihood $p$ similarly effective as improvements of signal quality $\rho$

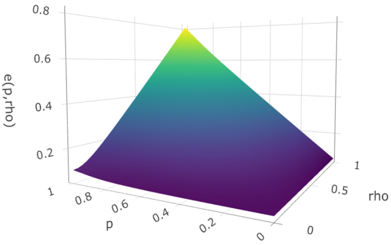
# Optimal investment with chance of noisy signal

## Solution (B.-Körber) (ctd):

This policy yields the indirect utility

$$v(t,x) = U\left(xe^{\left(r+e^0(\phi^0)+p\lambda\int_{\mathbb{R}} e^z(\phi^z)N_{0,1}(dz)\right)(T-t)}\right)$$

when used as of time $t \in [0, T]$.




- $e(0,\rho) = e^M(\phi^M)$: Merton investor's indifference mark up due to stock investment possibility
- $e(p,0) > 0$: jump alerts afford hedging opportunity
- improvements of alert likelihood $p$ similarly effective as improvements of signal quality $\rho$

The value of not being predictable.

# Part III

# Exogenous vs. endogenous jumps

# Proactive and reactive controls causing jumps

New aspect: Timing of shocks and interventions

So far: Controlled system only exhibits exogenous jumps and the control just determines our exposure to the ensuing shocks
$\rightsquigarrow$ *classical* control problem: Merton problem

But: What if in addition our controls can cause an endogenous jump in the system?
$\rightsquigarrow$ *singular* control problem: Merton with proportional transaction costs

Issue: What if such an endogenous jump coincides with an exogenous one? Is the control jump a reaction to an exogenous shock—or is it a proactive intervention in preparation to such a shock?

# Proactive and reactive controls causing jumps

New aspect: Timing of shocks and interventions

So far: Controlled system only exhibits exogenous jumps and the control just determines our exposure to the ensuing shocks
$\rightsquigarrow$ *classical* control problem: Merton problem

But: What if in addition our controls can cause an endogenous jump in the system?
$\rightsquigarrow$ *singular* control problem: Merton with proportional transaction costs

Issue: What if such an endogenous jump coincides with an exogenous one? Is the control jump a reaction to an exogenous shock—or is it a proactive intervention in preparation to such a shock?

Answer: It can be both—

# Proactive and reactive controls causing jumps

New aspect: Timing of shocks and interventions

So far: Controlled system only exhibits exogenous jumps and the control just determines our exposure to the ensuing shocks
$\rightsquigarrow$ *classical* control problem: Merton problem

But: What if in addition our controls can cause an endogenous jump in the system?
$\rightsquigarrow$ *singular* control problem: Merton with proportional transaction costs

Issue: What if such an endogenous jump coincides with an exogenous one? Is the control jump a reaction to an exogenous shock—or is it a proactive intervention in preparation to such a shock?

Answer: It can be both—and the proactive control jump can even take into account extra information about the exogenous jump when we use Meyer $\sigma$-fields!

# Singular control problem: Irreversible investment

- Classic problem: Dixit and Pindyck ('94), Bertola (1998), Merhi and Zervos ('07), Riedel and Su ('11), Ferrari ('15), Al Motairi and Zervos ('17), De Angelis et al. ('17) ...
- Consider target functional:

$$\tilde{V}(C) = \mathbb{E}\left[\int_{[0,\infty)} \tilde{P}_t \, dC_t - \int_{[0,\infty)} \rho_t(C_t) \, dR_t\right] \to \max_{C \geq c_0 \nearrow}$$

  $\tilde{P}$ discounted reward process, $\rho_t(c)$ risk penalty convex in $c$, $R$ risk assessment clock
- *Standard* assumptions: $\tilde{P}_t = e^{-rt} P_t$ for compound Poisson $P_t = p + \sum_{k=1}^{N_t} Y_k$; $\rho_t(c) = c^2/2$;$dR_t = e^{-rt} dt$:

$$\mathbb{E}\left[\int_{[0,\infty)} e^{-rt} P_t \, dC_t - \int_{[0,\infty)} \frac{1}{2}(C_t)^2 e^{-rt} \, dt\right] \to \max_{C \geq c_0 \nearrow \text{ in } \mathscr{P}}$$

# Singular control problem: Irreversible investment

- Classic problem: Dixit and Pindyck ('94), Bertola (1998), Merhi and Zervos ('07), Riedel and Su ('11), Ferrari ('15), Al Motairi and Zervos ('17), De Angelis et al. ('17) ...
- Consider target functional:

$$\tilde{V}(C) = \mathbb{E}\left[\int_{[0,\infty)} \tilde{P}_t \, dC_t - \int_{[0,\infty)} \rho_t(C_t)\, dR_t\right] \to \max_{C \geq c_0 \nearrow}$$

  $\tilde{P}$ discounted reward process, $\rho_t(c)$ risk penalty convex in $c$, $R$ risk assessment clock
- *Alternative* assumptions: $\tilde{P}_t = e^{-rt} P_t$ for compound Poisson $P_t = p + \sum_{k=1}^{N_t} Y_k$; $\rho_t(c) = c^2/2$; $dR_t = e^{-rt} dN_t$:

$$\mathbb{E}\left[\int_{[0,\infty)} e^{-rt} P_t \, dC_t - \int_{[0,\infty)} \frac{1}{2}(C_t)^2 e^{-rt}\, dN_t\right] \to \max_{C \geq c_0 \nearrow \text{ in } \Lambda}$$

- Proactive and reactive control: $C$ làdlàg required!
  Signal on large jumps: $\Lambda^\eta = \mathscr{P} \vee \sigma\left(\sum_{n=1}^{N_\cdot} Y_n 1_{\{|Y_n| \geq \eta\}}\right)$.

# Heuristics from first order conditions

First order conditions for optimality of $\hat{C}$:

$$\mathbb{E}\left[\tilde{P}_S \middle| \mathscr{F}_S^{\Lambda}\right] \leq \mathbb{E}\left[\int_{[S,\infty)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_t)dR_t \middle| \mathscr{F}_S^{\Lambda}\right]$$

with "$=$" holding true whenever it is optimal to intervene: $d\hat{C}_S > 0$

# Heuristics from first order conditions

First order conditions for optimality of $\hat{C}$:

$$\mathbb{E}\left[\tilde{P}_S \middle| \mathscr{F}_S^\Lambda\right] \leq \mathbb{E}\left[\int_{[S,\infty)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_t)dR_t \middle| \mathscr{F}_S^\Lambda\right]$$

with "$=$" holding true whenever it is optimal to intervene: $d\hat{C}_S > 0$
If optimal to intervene at $S$, then for any $T$ with $T > S$:

$$\begin{aligned}\mathbb{E}\left[\tilde{P}_S - \tilde{P}_T \middle| \mathscr{F}_S^\Lambda\right] &\geq \mathbb{E}\left[\int_{[S,T)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_t)dR_t \middle| \mathscr{F}_S^\Lambda\right] \\ &\geq \mathbb{E}\left[\int_{[S,T)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_S)dR_t \middle| \mathscr{F}_S^\Lambda\right]\end{aligned}$$

$\rightsquigarrow \widehat{C}_S \leq \operatorname{ess\,inf}_T \ell_{S,T}^\Lambda =: L_S^\Lambda$

# Heuristics from first order conditions

First order conditions for optimality of $\hat{C}$:

$$\mathbb{E}\left[\tilde{P}_S \middle| \mathscr{F}_S^\Lambda\right] \leq \mathbb{E}\left[\int_{[S,\infty)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_t)dR_t \middle| \mathscr{F}_S^\Lambda\right]$$

with "$=$" holding true whenever it is optimal to intervene: $d\hat{C}_S > 0$
If not optimal to intervene at $S$, then for next optimal intervention time $T_S$ we get

$$\begin{aligned}\mathbb{E}\left[\tilde{P}_S - \tilde{P}_{T_S} \middle| \mathscr{F}_S^\Lambda\right] &\leq \mathbb{E}\left[\int_{[S,T_S)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_t)dR_t \middle| \mathscr{F}_S^\Lambda\right] \\ &= \mathbb{E}\left[\int_{[S,T_S)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_S)dR_t \middle| \mathscr{F}_S^\Lambda\right]\end{aligned}$$

$\rightsquigarrow \ \widehat{C}_S \geq \ell^\Lambda_{S,T_S} \geq \operatorname{ess\,inf}_T \ell^\Lambda_{S,T} =: L^\Lambda_S$
In conjunction with '$\leq$' from before: $\widehat{C}_S = c_0 \vee \sup_{v\in[0,S]} L^\Lambda_v$

# A stochastic representation theorem

Theorem (B.-El Karoui (2004), B.-Besslich (2021+))

*Under suitable integrability and upper-semicontinuity assumptions, there exists* $L^\Lambda \in \Lambda$ *such that*

$$\mathbb{E}\left[\tilde{P}_S \middle| \mathscr{F}_S^\Lambda\right] = \mathbb{E}\left[\int_{[S,\infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v\in[S,t]} L_v^\Lambda\right) dR_t \middle| \mathscr{F}_S^\Lambda\right], \quad S \in \mathscr{S}^\Lambda.$$

*The maximal such* $L^\Lambda$ *is uniquely determined by*

$$L_S^\Lambda = \operatorname{essinf}_{T\in\mathscr{S}^\Lambda, T>S} \ell_{S,T}^\Lambda, \quad S \in \mathscr{S}^\Lambda,$$

*where for* $S < T$, $\ell_{S,T}^\Lambda \in \mathscr{F}_S^\Lambda$ *is defined by*

$$\mathbb{E}\left[\tilde{P}_S - \tilde{P}_T \middle| \mathscr{F}_S^\Lambda\right] = \mathbb{E}\left[\int_{[S,T)} \frac{\partial}{\partial c}\rho_t(\ell_{S,T}^\Lambda) dR_t \middle| \mathscr{F}_S^\Lambda\right]$$

*on* $\{\mathbb{P}\left[R_{T-} - R_{S-} > 0 \middle| \mathscr{F}_S^\Lambda\right] > 0\}$ *and* $\ell_{S,T}^\Lambda := \infty$ *elsewhere.*

# Explicit solution in the compound Poisson example

Let $\tilde{P}_t = e^{-rt} P_t$ with $P_t = p + \sum_{k=1}^{N_t} Y_k$ for Poisson $N$ with param. $\lambda$, i.i.d. $Y_k \in L^2$, $\mathbb{E} Y_k = m$; $dR_t = e^{-rt} dN_t$; $\rho_t(c) = \frac{1}{2} c^2$;

$$\Lambda = \Lambda^\eta := \mathscr{P} \vee \sigma\left( \sum_{k=1}^{N_\cdot} Y_k 1_{\{|Y_k| \geq \eta\}} \right) \quad \rightsquigarrow \text{Large jump alerts}$$

Probability of alert: $p(\eta) = \mathbb{P}[|Y_k| \geq \eta]$.

- $p(\eta) = 0$: no alerts, predictable case $\Lambda^\eta = \mathscr{P}$
- $p(\eta) = 1$: alerts for all jumps, optional case $\Lambda^\eta = \mathscr{O}$
- $p(\eta) \in (0, 1)$: Meyer case $\mathscr{P} \subsetneq \Lambda^\eta \subsetneq \mathscr{O}$

# Solution in the predictable case

In the case $p(\eta) = 0$, i.e. without alerts::

$$L_t^{\mathscr{P}} = a(P_{t-} - b), \quad t \in [0, \infty),$$

where the constants $a, b$ are given by

$$a := \frac{1}{\mathbb{E}[R_{\infty-}]} = \frac{r}{\lambda},$$

$$b := \sup_{0<T \text{ pred.}} \frac{\mathbb{E}\left[\mathrm{e}^{-rT} \sum_{k=1}^{N_T} Y_k\right]}{1 - \mathbb{E}\left[\mathrm{e}^{-rT}\right]} = \frac{\mathbb{E}\left[\int_{[0,\infty)} \left(\sup_{v \in [0,t]} P_{v-} - p\right) dR_t\right]}{\mathbb{E}[R_{\infty-}]}.$$

$\rightsquigarrow C^{\mathscr{P}} = c_0 \vee \sup_{0 \le v \le .} L_v^{\mathscr{P}}$ left-continuous with exclusively reactive jumps because jump times are totally inaccessible to controller

# Solution in the Meyer case

In the case $p(\eta) \in (0,1)$ with alerts for some, but not all jumps:

$$L_t^{\Lambda^\eta} = \begin{cases} 0, & P_t^\eta \geq b,\ |\Delta P_t^\eta| \geq \eta, \\ \frac{r}{\lambda}(P_t^\eta - b), & P_t^\eta \geq b,\ |\Delta P_t^\eta| < \eta, \\ \inf\limits_{\gamma^0 \in (0, B_0^\eta \cdot (b - P_t^\eta))} f_1^\eta(\gamma^0, 0, P_t^\eta) < 0, & P_t^\eta < b,\ |\Delta P_t^\eta| \geq \eta, \\ \inf\limits_{\gamma^1 \in (-B_1^\eta \cdot (b - P_t^\eta), 0)} f_0^\eta(0, \gamma^1, P_t^\eta) < 0, & P_t^\eta < b,\ |\Delta P_t^\eta| < \eta \end{cases}$$

where $P_t^\eta := P_{t-} + \Delta P_t 1_{\{|\Delta P_t| \geq \eta\}}$, $t \geq 0$,

$$f_\Delta^\eta = \frac{\left(1 - \mathbb{E}\left[\mathrm{e}^{-rT^\eta(\gamma^0,\gamma^1)}\right]\right) p - \mathbb{E}\left[\mathrm{e}^{-rT^\eta(\gamma^0,\gamma^1)} \sum_{k=1}^{N_{T^\eta(\gamma^0,\gamma^1)}} Y_k\right]}{\frac{\lambda}{r}\left(1 - \mathbb{E}\left[\mathrm{e}^{-rT^\eta(\gamma^0,\gamma^1)}\right]\right) - \mathbb{E}\left[\mathrm{e}^{-rT^\eta(\gamma^0,\gamma^1)} 1_{\{|\Delta P_{T^\eta(\gamma^0,\gamma^1)}| \geq \eta\}}\right] + \Delta},$$

$$T^\eta(\gamma^0,\gamma^1) = \inf\Big\{t \in \{^{\Lambda^\eta} N > 0\} \;\Big|\; \big(|\Delta P_t| < \eta \text{ and } P_{t-} - p \geq \gamma^0\big) \text{ or } \big(|\Delta P_t| \geq \eta \text{ and } P_t - p \geq \gamma^1\big)\Big\}.$$

# Solution in the optional case

In the optional case with alerts for all jumps $p(\eta) = 1$:

$$L_t^{\mathscr{O}} = \begin{cases} 0, & P_t \geq b,\ |\Delta P_t| > 0, \\ \frac{r}{\lambda}(P_t - b), & P_t \geq b,\ \Delta P_t = 0, \\ \frac{r}{\lambda + r}(b - P_t), & P_t < b,\ |\Delta P_t| > 0, \\ \inf\limits_{\gamma \in (-\infty, 0)} f(\gamma, P_t) < 0, & m\frac{\lambda}{r} \leq P_t < b,\ \Delta P_t = 0, \\ -\infty, & P_t < m\frac{\lambda}{r},\ \Delta P_t = 0. \end{cases}$$

where

$$f(\gamma, p) := \frac{\left(1 - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]\right) p - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)} \sum_{k=1}^{N_{T(\gamma)}} Y_k\right]}{\frac{\lambda}{r}\left(1 - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]\right) - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]},$$

$$T(\gamma) := \inf\left\{ t \in \{N > 0\} \,\middle|\, |\Delta P_t| > 0 \text{ and } P_t - p \geq \gamma \right\}.$$

# Solution in the optional case

In the optional case with alerts for all jumps $p(\eta) = 1$:

$$L_t^{\mathscr{O}} = \begin{cases} 0, & P_t \geq b,\ |\Delta P_t| > 0, \\ \frac{r}{\lambda}(P_t - b), & P_t \geq b,\ \Delta P_t = 0, \\ \frac{r}{\lambda + r}(b - P_t), & P_t < b,\ |\Delta P_t| > 0, \\ \inf\limits_{\gamma \in (-\infty, 0)} f(\gamma, P_t) < 0, & m\frac{\lambda}{r} \leq P_t < b,\ \Delta P_t = 0, \\ -\infty, & P_t < m\frac{\lambda}{r},\ \Delta P_t = 0. \end{cases}$$

where

$$f(\gamma, p) := \frac{\left(1 - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]\right) p - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)} \sum_{k=1}^{N_{T(\gamma)}} Y_k\right]}{\frac{\lambda}{r}\left(1 - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]\right) - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]},$$

$$T(\gamma) := \inf\left\{ t \in \{N > 0\} \,\middle|\, |\Delta P_t| > 0 \text{ and } P_t - p \geq \gamma \right\}.$$

**Observation:** $L_t^{\mathscr{P}} \xleftarrow[p(\eta)\to 1]{} L^{\Lambda^\eta} \xrightarrow[p(\eta)\to 0]{} L_t^{\mathscr{O}}$

# Illustration

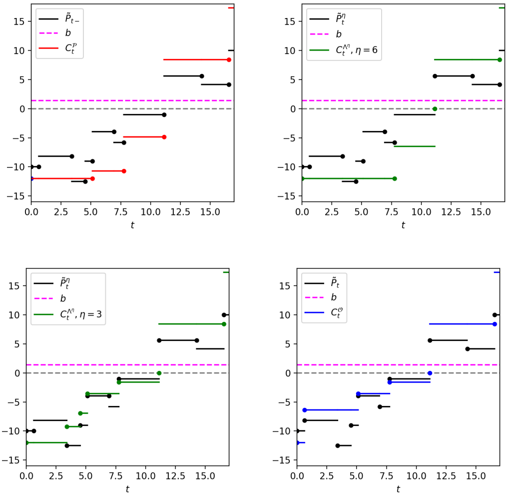



**Figure:** $P^{\eta}$ (black), $b$ (Magenta) and optimal controls for $\eta = 0$ (blue, optional), $\eta = 3$, $\eta = 6$ (green) and $\eta = \infty$ (red, predictable). The dots indicate the processes' value at their jump times.

# Illustration

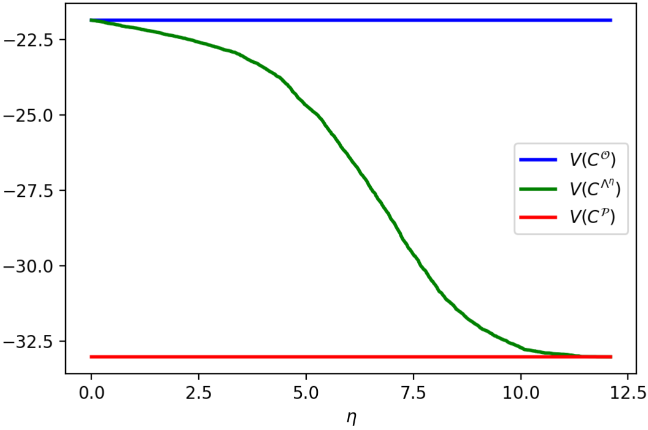



**Figure:** The value of not being predictable.

# References


- Lenglart: Tribus de Meyer et théorie des processus, Springer LNM 784, '80
- El Karoui: Les aspects probabilistes du contrôle stochastique, Springer LNM 876, '81
- Campi, Schachermayer: A super-replication theorem in Kabanov's model of transaction costs. Finance Stoch., '06
- Czichowsky, Schachermayer: Duality theory for portfolio optimisation under transaction costs. Ann. Appl. Probab., '16
- with Besslich: Modelling information flows by Meyer-$\sigma$-fields in the singular stochastic control problem of irreversible investment, Ann. Appl. Probab., '20
- with Besslich: On a Stochastic Representation Theorem for Meyer-measurable Processes and its Applications in Stochastic Optimal Control and Optimal Stopping, Annales de l'IHP, '21+
- with Körber: Merton's optimal investment problem with jump signals (working title), in preparation

# Conclusion and Outlook

- continuous-time information modeling most flexibly done via Meyer $\sigma$-fields
- allows for modeling signals on jump events and for strategies to act on them: interpolation between predictable and optional information flow
- Merton's optimal investment problem with signals on jumps: there is value in not being predictable in classical control problems
- Irreversible investment: explicit solution in compound Poisson setting with jump size dependent alerts; pro-active and re-active interventions in a singular control with jump signals

# Conclusion and Outlook

- continuous-time information modeling most flexibly done via Meyer $\sigma$-fields
- allows for modeling signals on jump events and for strategies to act on them: interpolation between predictable and optional information flow
- Merton's optimal investment problem with signals on jumps: there is value in not being predictable in classical control problems
- Irreversible investment: explicit solution in compound Poisson setting with jump size dependent alerts; pro-active and re-active interventions in a singular control with jump signals
- ongoing: duality in model with transient price impact, more tools for dynamic programming, further applications

**Thank you very much!**

# Optimal stopping

$$\mathbb{E}\left[X_T + \int_{[0,T)} g_t(\ell)\mu(dt)\right] \to \max_{T \in \mathscr{S}^\Lambda}$$

- Reward upon stopping: $\Lambda$-measurable $X$ of class $D^\Lambda$
- running reward: $g = g_t(\omega, \ell)$: strictly & continuously increasing from $-\infty$ to $+\infty$ in parameter $\ell \in (-\infty, +\infty)$
- payment clock: $\mu(dt)$, possibly random and with atoms with $\mathbb{E}[\int_{[0,\infty)} g_t(\ell)\mu(dt)] < \infty$ for any $\ell$

Questions: Existence and construction of optimal stopping time?
Note: Inevitable lack of upper-semi-continuity if $\mu$ has atoms!

# Optimal stopping

$$\mathbb{E}\left[X_\tau + \int_{[0,\tau)} g_t(\ell)\mu(dt)\right] \to \max_{\tau \in \mathscr{S}^{\Lambda,\text{div}}}$$

- Reward upon stopping: $\Lambda$-measurable $X$ of class $D^\Lambda$
- running reward: $g = g_t(\omega, \ell)$: strictly & continuously increasing from $-\infty$ to $+\infty$ in parameter $\ell \in (-\infty, +\infty)$
- payment clock: $\mu(dt)$, possibly random and with atoms with $\mathbb{E}[\int_{[0,\infty)} g_t(\ell)\mu(dt)] < \infty$ for any $\ell$

Questions: Existence and construction of optimal stopping time?
Note: Inevitable lack of upper-semi-continuity if $\mu$ has atoms!
**El Karoui**'s relaxation to divided stopping times considers

$$\tau = (T, H^l, H, H^r) \text{ with } X_\tau = {}^*X_T 1_{H^l} + X_T 1_H + X^*_T 1_{H^r}$$

# Universal stopping signal

**Theorem**

*Let a "suitably u.s.c." $X$ be represented by $L$ as*

$$X_S = \mathbb{E}\left[\int_{[S,\infty)} g_t(\sup_{v\in[S,t]} L_v)\mu(dt)\middle|\mathscr{F}_S^\Lambda\right], \quad S \in \mathscr{S}^\Lambda.$$

*Then both*

$$\tau^{\geq\ell} = (T^{\geq\ell}, \varnothing, H^{\geq\ell}, \Omega\backslash(H^{\geq\ell})) \text{ and } \tau^{>\ell} = (T^{>\ell}, \varnothing, H^{>\ell}, \Omega\backslash(H^{>\ell}))$$

*given, respectively, by*

$$T^{\geq\ell} = \inf\{t \geq 0 \mid \sup_{[0,t]} L \geq \ell\}, \quad H^{\geq\ell} = \{L_{T^{\geq\ell}} \geq \ell\},$$
$$T^{>\ell} = \inf\{t \geq 0 \mid \sup_{[0,t]} L > \ell\}, \quad H^{>\ell} = \{L_{T^{>\ell}} > \ell\},$$

*are optimal for the stopping problem with parameter $\ell$.*

# Illustration

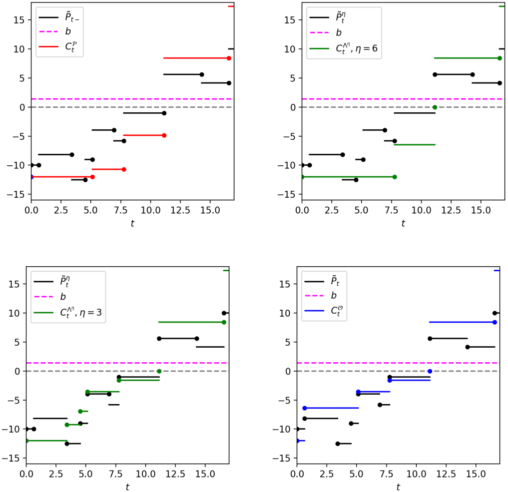



**Figure:** $\tilde{P}^{\eta}$ (black), $b$ (Magenta) and optimal controls for $\eta = 0$ (blue, optional), $\eta = 3$, $\eta = 6$ (green) and $\eta = \infty$ (red, predictable). The dots indicate the processes' value at their jump times.