Mean Field Game Master Equations with Monotonicity and Anti-monotonicity Conditions in Displacement Sense

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Outline

N-player games

- A large system : $i = 1, \cdots, N$,

$$X_t^{i,\alpha^i} = x_i + \int_0^t \alpha_s^i(X_s^{i,\alpha^i})ds + B_s^i; \quad \mu_s^{N,\vec{\alpha}} := \frac{1}{N}\sum_{i=1}^N \delta_{X_s^{i,\alpha^i}}.$$

- $J_i(0,\vec{x},\vec{\alpha}) := \boldsymbol{E}\Big[G(X_T^{i,\alpha^i}, \mu_T^{N,\vec{\alpha}}) + \int_0^T F(X_s^{i,\vec{\alpha}}, \mu_s^{N,\vec{\alpha}}, \alpha_s^i)ds\Big]$

- Nash equilibrium $\vec{\alpha}^*$:

$$J_i(0,\vec{x},\vec{\alpha}^*) \le J_i(0,\vec{x},\vec{\alpha}^{*,-i},\alpha^i), \quad \forall \alpha^i, \forall i.$$

- The goal : $N \to \infty$?

Mean field games

- The "others" : everybody using the same α,

$$X_t^{\xi,\alpha} = \xi + \int_0^t \alpha_s(X_s^{\xi,\alpha})ds + B_s; \quad \mu_s^\alpha := \mathcal{L}_{X_s^{\xi,\alpha}}.$$

- The individual player : using α',

$$X_t^{x,\alpha'} = x + \int_0^t \alpha'_s(X_s^{x,\alpha'})ds + B_s;$$

$$J(0,\xi,\alpha;x,\alpha') := \boldsymbol{E}\Big[G(X_T^{x,\alpha'},\mu_T^\alpha) + \int_0^T F(X_s^{x,\alpha'},\mu_s^\alpha,\alpha'_s)ds\Big].$$

Mean field equilibrium

- Mean field equilibrium α^* :

$$J(0,\xi,\alpha^*;x,\alpha^*) \leq J(0,\xi,\alpha^*;x,\alpha'), \quad \forall \alpha', \ \forall x.$$

- When MFE α^* is unique, introduce the value function

$$V(t,x,\mu) := J(t,\xi,\alpha^*;x,\alpha^*).$$

Our goal is to study the master equation for V.

- Caines-Huang-Malhame (2006), Lasry-Lions (2007) ;

Cardaliaguet (2010), Carmona-Delarue (2018)

The master equation

The global wellposedness of the master equation required :

- Sufficiently smooth data F, G
- F separable : $F(x, \mu, \alpha) = F_0(x, \mu) + F_1(x, \alpha)$.
- G, F_0 monotone (in Lasry-Lions sense or in displacement sense)

- The literature for global solution :
  - ◇ Buckdahn-Li-Peng-Rainer (2017) ;
  - ◇ Chassagneux-Crisan-Delarue (2014), Carmona-Delarue (2018), Cardaliaguet-Delarue-Lasry-Lions (2019), Mou-Z. (2019)
  - ◇ Gangbo-Meszaros (2020), Bensoussan-Graber-Yam (2020)
  - ◇ Bayraktar-Cohen (2018), Bertucci-Lasry-Lions (2019), Bayraktar-Cecchin-Cohen-Delarue (2019)

Our main contributions

- Non-separable F
- New types of monotonicity and anti-monotonicity conditions
- A new approach to find the appropriate conditions

- Gangbo-Meszaros-Mou-Z. (2021), Mou-Z. (2021 ?)

Our strategy

Monotonicity of data

Monotonicity of V

Global wellposedness of master equation

Our strategy (cont.)

Anti-Monotonicity of data

Anti-Monotonicity of V

Global wellposedness of master equation

Outline of the talk

- Stochastic calculus on Wasserstein space
- Derivation of our (anti-)monotonicity condition
- The global wellposedness

- For simplicity
  - ◇ $d = 1$
  - ◇ No common noise

Outline

Wasserstein derivatives

- $\mathcal{P}_2 = \mathcal{P}_2(\mathbb{R})$, Wasserstein distance W_2
- For $U : \mathcal{P}_2 \to \mathbb{R}$, linear derivative $\delta_\mu U : \mathcal{P}_2 \times \mathbb{R} \to \mathbb{R}$
  - ◇ $\lim_{\varepsilon \to 0} \frac{U(\mu + \varepsilon(\nu - \mu)) - U(\mu)}{\varepsilon} = \int_{\mathbb{R}} \delta_\mu U(\mu, x)[\nu(dx) - \mu(dx)]$
  - ◇ Wasserstein derivative $\partial_\mu U(\mu, x) = \partial_x \delta_\mu U(\mu, x)$
- For $U : (t, x, \mu) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2 \to \mathbb{R}$,
  - ◇ $\partial_t U, \partial_x U, \partial_{xx} U$ are standard
  - ◇ $\delta_\mu U(t, x, \mu, \tilde{x})$ and higher order derivatives in obvious sense

Ito formula

- Let $U(t,x,\mu)$ be smooth, and $dX_t = b_t dt + \sigma_t dB_t$

$$dU(t,X_t,\mathcal{L}_{X_t}) = \partial_t U dt + \partial_x U dX_t + \frac{1}{2}\partial_{xx} U \sigma_t^2 dt$$
$$+\tilde{\boldsymbol{E}}\Big[\frac{1}{2}\partial_{\tilde{x}\mu} U(t,X_t,\mathcal{L}_{X_t},\tilde{X}_t)\tilde{\sigma}_t^2 + \partial_\mu U(t,X_t,\mathcal{L}_{X_t},\tilde{X}_t)\tilde{b}_t \Big| \mathcal{F}_t^X\Big] dt$$

$\diamond$ $(\tilde{X},\tilde{b},\tilde{\sigma})$ is an independent copy of (X,b,σ)

$\diamond$ Buckdahn-Li-Peng-Rainer (2017), Chassagneux-Crisan-Delarue (2014)

Hamiltonian

- MFG :

$$dX_t = \alpha_t^* dt + dB_t;$$
$$V_0 = \boldsymbol{E}\Big[G(X_T, \mathcal{L}_{X_T}) + \int_0^T F(X_t, \mathcal{L}_{X_t}, \alpha_t^*)dt\Big]$$

- Hamiltonian $H(x, \mu, p) := \inf_a \Big[ap + F(x, \mu, a)\Big]$
  - ◇ $\partial_{pp} H < 0$
  - ◇ $a^* = \partial_p H(t, x, \mu, p)$

- Separability :

$$F = F_0(x, \mu) + F_1(x, a) \Longrightarrow H = F_0(x, \mu) + H_1(x, p)$$

Master equation

- DPP + Ito formula $\Longrightarrow$ PDE
- Master equation for $V(t,x,\mu)$

$$0 = \partial_t V + \frac{1}{2}\partial_{xx}V + \partial_x V \ \partial_p H(x,\mu,\partial_x V(t,x,\mu))$$
$$+\tilde{\boldsymbol{E}}\Big[\frac{1}{2}\partial_{\tilde{x}\mu}V(t,x,\mu,\tilde{\xi}) + \partial_\mu V(t,x,\mu,\tilde{\xi})\ \partial_p H(\tilde{\xi},\mu,\partial_x V(t,\tilde{\xi},\mu))\Big];$$
$$V(T,x,\mu) = G(x,\mu).$$

◇ Optimal control : $\alpha_t^* = \partial_p H(X_t,\mathcal{L}_{X_t},\partial_x V(t,X_t,\mathcal{L}_{X_t}))$

◇ Non-local equation, comparison principle fails

Monotonicity conditions

- Lasry-Lions monotonicity condition

$$\boldsymbol{E}\Big[G(\xi_1,\mathcal{L}_{\xi_1})+G(\xi_2,\mathcal{L}_{\xi_2})-G(\xi_1,\mathcal{L}_{\xi_2})-G(\xi_2,\mathcal{L}_{\xi_1})\Big]\geq 0;$$

or equivalently $\tilde{\boldsymbol{E}}\Big[\partial_{x\tilde{x}}\delta_\mu G\big(\xi,\mathcal{L}_\xi,\tilde{\xi}\big)\ \eta\tilde{\eta}\Big]\geq 0,\quad \forall\xi,\eta$

- Displacement monotonicity condition

$$\tilde{\boldsymbol{E}}\Big[\partial_{x\tilde{x}}\delta_\mu G\big(\xi,\mathcal{L}_\xi,\tilde{\xi}\big)\ \eta\tilde{\eta}+\partial_{xx}G(\xi,\mathcal{L}_\xi)\eta^2\Big]\geq 0,\quad \forall\xi,\eta \qquad (1)$$

- For potential games : $G(x,\mu)=\delta_\mu\mathcal{G}(\mu,x)$ for some $\mathcal{G}:\mathcal{P}_2\to\mathbb{R}$
  - ◇ Lasry-Lions monotonicity means : $\mu\in\mathcal{P}_2\to\mathcal{G}(\mu)$ is convex
  - ◇ Displacement monotonicity means : $\xi\in\mathbb{L}^2\to\mathcal{G}(\mathcal{L}_\xi)$ is convex

A motivating example

- Lasry-Lions : $G(x,\mu) = (x - m_\mu)^2$, where $m_\mu := \int x\mu(dx)$

$$\partial_{x\tilde{x}}\delta_\mu G(x,\mu,\tilde{x}) = -2,$$

$$\tilde{\boldsymbol{E}}\left[\partial_{x\tilde{x}}\delta_\mu G\left(\xi,\mathcal{L}_\xi,\tilde{\xi}\right)\ \eta\tilde{\eta}\right] = -2|\boldsymbol{E}[\eta]|^2 \leq 0.$$

- Carmona-Cooney-Graves-Lauriere :

$$G(x,\mu) = (x - m_\mu)^2 + (x - x_0)^2$$

- Displacement anti-monotonicity :

$$G(x,\mu) = (x - m_\mu)^2 - (x - x_0)^2$$

Outline

Characteristics

- Given ξ, η,

$$X_t = \xi + \int_0^t \partial_p H(X_s, \mu_s, \partial_x V(s, X_s, \mu_s))ds + B_t, \;\; \mu_t := \mathcal{L}_{X_t};$$

$$\delta X_t = \eta + \int_0^t \Big[H_{px}(X_s)\delta X_s + \frac{1}{2}\tilde{\boldsymbol{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t, \tilde{X}_t)\delta \tilde{X}_t] + H_{pp}(X_s)N_s \Big] ds,$$

$$N_t := \tilde{\boldsymbol{E}}_{\mathcal{F}_t}[\partial_{x\mu} V(X_t, \tilde{X}_t)\delta \tilde{X}_t] + \partial_{xx} V(X_t)\delta X_t + \frac{\tilde{\boldsymbol{E}}_{\mathcal{F}_t}[H_{p\mu}(X_t, \tilde{X}_t)\delta \tilde{X}_t]}{2H_{pp}(X_t)}$$

$\diamond$ $\delta X_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[X_t^{\xi+\varepsilon\eta} - X_t^{\xi}]$

The main idea

- For Lasry-Lions monotonicity, want

$$I'_{LL}(t) \le 0, \qquad I_{LL}(t) := \tilde{\boldsymbol{E}}\Big[\partial_{x\mu} V(t, X_t, \mu_t, \tilde{X}_t)\ \delta X_t \delta \tilde{X}_t\Big].$$

- Notice that $V(T, \cdot) = G$,

G L-L mon $\Leftrightarrow I_{LL}(T) \ge 0 \ \Rightarrow\ I_{LL}(0) \ge 0 \Leftrightarrow V(0, \cdot, \cdot)$ L-L mon

- Ito formula + master equation to analyze $I'_{LL}(t)$

The key calculation

- The non-separable case

$$I'_{LL}(t) = \tilde{\boldsymbol{E}}\Big[H_{pp}(X_t)\Big|\tilde{\boldsymbol{E}}_{\mathcal{F}_t}\Big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t + \frac{H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t}{2H_{pp}(X_t)}\Big]\Big|^2$$

$$-\partial_{xx}V(X_t)H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\delta X_t$$

$$-H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\delta X_t - \frac{\Big|\tilde{\boldsymbol{E}}_{\mathcal{F}_t}\big[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\big]\Big|^2}{4H_{pp}(X_t)}\Big].$$

- The separable case : $H = H_0(x,\mu) + H_1(x,p)$, and $H_{p\mu} = 0$,

$$I'_{LL}(t) = \tilde{\boldsymbol{E}}\Big[H_{pp}(X_t)\Big|\tilde{\boldsymbol{E}}_{\mathcal{F}_t}\Big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t\Big]\Big|^2 - H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\delta X_t\Big]$$

◇ Need : H_0 is Lasry-Lions monotone

Displacement monotonicity

$$I_D'(t) := \frac{d}{dt}\tilde{\boldsymbol{E}}\Big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta X_t\delta\tilde{X}_t + \partial_{xx}V(X_t)|\delta X_t|^2\Big]$$

$$= \tilde{\boldsymbol{E}}\Bigg[H_{pp}(X_t)\Big|\tilde{\boldsymbol{E}}_{\mathcal{F}_t}\Big[V_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t + V_{xx}(X_t)\delta X_t + \frac{H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t}{2H_{pp}(X_t)}\Big]\Big|^2$$

$$-H_{x\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\delta X_t - H_{xx}(X_t)|\delta X_t|^2 - \frac{\Big|\tilde{E}_{\mathcal{F}_t}\Big[H_{p\mu}(X_t,\tilde{X}_t)\delta\tilde{X}_t\Big]\Big|^2}{4H_{pp}(X_t)}\Bigg].$$

- Our monotonicity condition

$$\tilde{\boldsymbol{E}}\Big[H_{x\mu}(\xi,\tilde{\xi})\eta\tilde{\eta} + H_{xx}(\xi)\eta^2 + \frac{\Big|\tilde{\boldsymbol{E}}_{\mathcal{F}_t}\big[H_{p\mu}(\xi,\tilde{\xi})\tilde{\eta}\big]\Big|^2}{4H_{pp}(\xi)}\Big] \geq 0 \qquad (2)$$

- The separable case : $H = H_0(x,\mu) + H_1(x,p)$, and $H_{pu} = 0$,
  - ◇ Need : H_0 is displacement monotone

Displacement anti-monotonicity

- Recall

$$I_D(t) := \tilde{\boldsymbol{E}}\Big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta X_t\delta\tilde{X}_t + \partial_{xx}V(X_t)|\delta X_t|^2\Big]$$

- Denote

$$J_D(t) := \boldsymbol{E}\Big[\Big|\tilde{\boldsymbol{E}}_{\mathcal{F}_T}\big[\partial_{x\mu}V(X_t,\tilde{X}_t)\delta\tilde{X}_t\big]\Big|^2 + \Big|\partial_{xx}V(X_t)\delta X_t\Big|^2\Big] \geq 0.$$

- Anti-monotonicity : for some appropriate constant $c > 0$,

$$I_D(t) + cJ_D(t) \leq 0 \quad \text{or equivalently} \quad I_D(t) \leq -cJ_D(t) \leq 0.$$

- $I_D(t)$ and $J_D(t)$ involve different orders of $\partial_{xx}V$ and $\partial_{x\mu}V$, so we need precise estimates on their bounds, in particular, $\partial_{xx}V < 0$.

Propagation of anti-monotonicity

• Assume G satisfies : $I_D(T) + cJ_D(T) \leq 0$

• Assume H satisfies certain condition which gives us appropriate estimates on $\partial_{xx} V$ (and $\partial_{x\mu} V$)

• The crucial estimation :

$$I'_D(t) + cJ'_D(t) \geq 0 \quad \Longrightarrow \quad I_D(t) + cJ_D(t) \leq 0, \ \forall t.$$

• In the separable case, we may consider Lasry-Lions anti-monotonicity in the same manner :

$$\tilde{\boldsymbol{E}}\left[\partial_{x\mu} V(X_t, \tilde{X}_t)\delta X_t \delta \tilde{X}_t + \left|\tilde{\boldsymbol{E}}_{\mathcal{F}_T}\left[\partial_{x\mu} V(X_t, \tilde{X}_t)\delta \tilde{X}_t\right]\right|^2\right] \leq 0.$$

Outline

Assumptions

- (Areg) : H, G are sufficiently smooth and $H_{pp} \le -c_0 < 0$
- (AG) : G satisfies diaplacement monotonicity condition (1)
- (AH) : H satisfies our monotonicity condition (2)
- (AV) : V is a sufficiently smooth solution of the master equation

• The conditions in the anti-monotonicity case are more involved. We skip them here. Once we obtain the propagation of the anti-monotonicity, the remaining proof of the global wellposedness is the same as the monotone case presented at below.

Propagation of displacement monotonicity

Theorem 1

Under (Areg), (AG), (AH), and (AV), then V satisfies the displacement monotonicity condition (1).

Uniform Lipschitz continuity of V

Theorem 2

Under (Areg) and (AV), if V satisfies the Lasry-Lions monotonicity condition or the displacement monotonicity condition, then V is uniformly Lipschitz continuous in x and μ

- In Theorem 2, we do not assume monotonicity condition directly on H.

Global wellposedness

Theorem 3

Under (Areg), (AG), and (AH), then the master equation has a unique classical solution V

- Local classical solution + Uniform Lipschitz continuity

  $\Longrightarrow$ Global classical solution

A final remark

The monotonicity condition is for the

- uniqueness of mean field equilibrium
- existence of global solution
- not for the uniqueness of global solution

Thank you very much for your attention !