Entropy Martingale Optimal Transport and Nonlinear Pricing-Hedging Duality

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Bachelier Finance Society One World Seminar
September 30, 2021

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Introduction

The objective of this work is to develop a duality for an Entropy Martingale Optimal Transport (EMOT) problem.

- In EMOT we follow Liero et al. in "Optimal entropy-transport problems and a new Hellinger-Kantorovic distance between positive measures", Invent. math. 2018, but we add the constraint, typical of Martingale Optimal Transport (MOT) theory, that the infimum of the cost functional is taken over martingale probability measures.

Introduction

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- In the dual optimization problem the objective functional, given via Fenchel coniugacy, is not any more linear, as in OT or in MOT. This has a clear financial interpretation as a non linear subhedging value.

Robust Pricing-Hedging Duality

- One key assumption of robust finance is that the marginals $(\widehat{Q}_0, \widehat{Q}_1, ..., \widehat{Q}_T)$ of the underlying price process X are known .
- These marginals could be inferred (Breeden Litzenberger (1978), Hobson (1998)) from the knowledge of a "very large" number of prices of plain vanilla options.

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- In EMOT the marginals are not any more fixed a priori, because we may not have sufficient information to detect them with enough accuracy, for example if:
  - there are not sufficiently many traded options and is impossible to extract precisely the marginals via Breeden and Litzenberger;
  - the exact prices of the options might be unknown (market impact effects).

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  - there are not sufficiently many traded options and is impossible to extract precisely the marginals via Breeden and Litzenberger;

  - the exact prices of the options might be unknown (market impact effects).
- So in EMOT the infimum is taken over all martingale probability measures, but those that are far from some estimate $(\widehat{Q}_0, \widehat{Q}_1, \ldots \widehat{Q}_T)$ are penalized.
- This is a key difference with the literature on this topic, as discussed in the sequel.

Pathwise setting: no reference prob. meas. P

Classical Optimal Transport (the sublinear case)

$$\inf_{Q \in \Pi(Q_1, Q_2)} E_Q[c] = \sup_{\varphi + \psi \leqslant c} \left(\mathbb{E}_{Q_1}[\varphi] + \mathbb{E}_{Q_2}[\psi] \right)$$

No Market and linear pricing

(Monge, Kantorovic, Rachev, Ruschendorf, Villani ...)

Pathwise setting: no reference prob. meas. P

Entropy Optimal Transport (the convex case)

$$\inf_{Q \in \mathrm{Meas}(\Omega)} \left(E_Q[c] + \mathcal{D}^U(Q) \right) = \sup_{\varphi + \psi \leqslant c} U(\varphi, \psi)$$

No Market and non linear pricing and penalization

(Liero et al., Invent. math., 2018)

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Classical Martingale Optimal Transport (the sublinear case)

$$\inf_{Q \in \mathrm{Mart}(Q_1, Q_2)} E_Q[c] = \sup_{[\varphi,\psi] \in \mathrm{Sub}(c)} (\mathbb{E}_{Q_1}[\varphi] + \mathbb{E}_{Q_2}[\psi])$$

Trading in the market and linear pricing (Beiglböck et al. 2013, Davis et al. 2014, Galichon et al. 2014 ...)

Pathwise setting: no reference prob. meas. P

Entropy Optimal Transport (the convex case)

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No Market and non linear pricing and penalization

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Entropy Martingale Optimal Transport (the convex case)

$$\inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[c] + \mathcal{D}^U(Q) \right) = \sup_{[\varphi, \psi] \in \mathrm{Sub}(c)} S^U(\varphi, \psi)$$

Trading in the market and non linear pricing and penalization

What is S^U ?

Literature

Optimal Transport theory proved to be a very powerful tool to prove pathwise pricing hedging duality results in discrete and continuous time:

Beiglbock et al. (2013), Tan and Touzi (2013), Davis et al. (2014), Galichon et al. (2014), Dolinksi and Soner (2014) (2015), Henry-Labordere et al. (2016), Hou and Obloj (2018), Bartl et al. (2019), Wiesel (2019), Guo and Obloj (2019), Cheridito et al. (2020).

Not to mention: Weak Optimal Transport, Causal Optimal Transport, stability of MOT...

The addition of an entropic term to optimal transport problems was popularized by Cuturi (2013), Cuturi and Peyre' (2019); and more recently, Nutz and Wiesel (2021), Bernton al. (2021). This, however, is an approach different from our EMOT problem.

On Weak Martingale Optimal Entropy Transport: Chung andTrinh (2021).

Pathwise setting: no reference prob. meas. P

We take $\Omega = K_0 \times \cdots \times K_T \subseteq \mathbb{R}^{T+1}$ and the stock X is given by projections on factors.

$$\mathrm{Mart}(\Omega) := \{\text{Martingale probability measures for } X\},$$

The class of compatible arbitrage-free pricing measures is given by

$$\mathcal{M}(\widehat{Q}_0, \widehat{Q}_1, \ldots \widehat{Q}_T) := \left\{ Q \in \mathrm{Mart}(\Omega) \mid X_t \sim_Q \widehat{Q}_t \text{ for each } t = 0, \ldots, T \right\}.$$

and we introduce admissible integrands and stochastic integrals as

$$\mathcal{H} := \{\Delta = [\Delta_0, \ldots, \Delta_{T-1}] \mid \Delta_t \in \mathcal{C}_b(K_0 \times \cdots \times K_t; \mathbb{R})\}$$

$$\mathcal{I} := \left\{ I^\Delta(x) = \sum_{t=0}^{T-1} \Delta_t(x_0, \ldots, x_t)(x_{t+1} - x_t) \mid \Delta \in \mathcal{H} \right\}$$

Robust Pricing-Hedging Duality

In this framework, the sub-hedging duality for the contingent claim $c : \Omega \to (-\infty, +\infty]$, obtained in Beiglböck et al. "Model-independent bounds for option prices - a mass transport approach", F&S (2013), takes the form:

$$\inf_{Q \in \mathcal{M}(\widehat{Q}_0, \widehat{Q}_1, \ldots \widehat{Q}_T)} E_Q[c] = \tag{1}$$

$$= \sup \left\{ \sum_{t=0}^{T} E_{\widehat{Q}_t}[\varphi_t] \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi_t(x_t) + I^{\Delta}(x) \leqslant c(x) \ \forall x \in \Omega \right\}, \tag{2}$$

where the (2) is known as the **robust subhedging price** of c and (1) represents the dual problem in the financial application, but is typically the primal problem in **Martingale Optimal Transport** (MOT)

Towards EMOT

The **Entropy Optimal Transport** (EOT) problem (Liero et al. 2018) takes the form:

$$\inf_{\mu \in \mathrm{Meas}(\Omega)} \left(E_Q\left[c\right] + \sum_{t=0}^{T} \mathcal{D}_{v_t^*, \widehat{Q}}(\mu_t) \right),$$

where $\mathrm{Meas}(\Omega)$ is the set of all positive finite measures μ on Ω, and $\mathcal{D}_{v_t^*, \widehat{Q}}(\mu_t)$ is a divergence functional.

A similar approach has also been developed by Pennanen-Perkkiö 2017.

- The key idea in EOT is to relax the marginal constraint $X_t \sim_Q \widehat{Q}_t$ in OT using a penalization term.
- We follow a similar idea, replacing the marginal constraints in MOT with penalization terms and keeping computing the inf over the set of (martingale) probability measures.

The marginals are not kept fixed

This yields the EMOT problem:

$$\mathfrak{D}(c) := \inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[c] + \sum_{t=0}^{T} \mathcal{D}_{v_t^*, \widehat{Q}_t}(Q_t) \right),$$

where

$$\mathcal{D}_{v_t^*, \widehat{Q}_t}(Q_t) := \int_{K_t} v_t^* \left(\frac{\mathrm{d}Q_t}{\mathrm{d}\widehat{Q}_t} \right) \mathrm{d}\widehat{Q}_t, \text{ if } Q_t \ll \widehat{Q}_t, \quad \text{for } Q_t \in \mathrm{Prob}(K_t),$$

and $v_t^*(y) = \sup_{x_t \in \mathbb{R}} \{x_t y - v_t(y)\} = -u_t^*(y)$, for $v_t(y) := -u_t(-y)$.

- if $v_t^* = \delta_{\{1\}}$ (i.e. if $u_t(x) = x$) then $\mathcal{D}_{v_t^*, \widehat{Q}_t} = \delta_{\widehat{Q}_t}$ and we recover MOT.
- In the sequel we will consider penalization terms $\mathcal{D}_t$ more general than those having the divergence formulation $\mathcal{D}_{v_t^*, \widehat{Q}_t}$.

On EMOT compared with MOT with relaxation term

As mentioned, the EMOT problem we will consider has the form:

$$\mathfrak{D}(c) := \inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[c] + \mathcal{D}(Q) \right).$$

If we take penalization terms $\mathcal{D}$ of the form

$$\mathcal{D}(Q) := \sum_{t=0}^{T} \delta_{\widehat{Q}_t}(Q)$$

we obtain the classical MOT problem. If we take penalizations of the form

$$\mathcal{D}(Q) := \sum_{t=0}^{T} \delta_{\widehat{Q}_t}(Q_t) + \widehat{\mathcal{D}}(Q)$$

form some (entropic) penalizations $\widehat{\mathcal{D}}_t$, we obtain the MOT problem with entropic relaxation:

$$\mathfrak{D}(c) := \inf_{Q \in \mathcal{M}(\widehat{Q}_0, \widehat{Q}_1, \dots \widehat{Q}_T)} \left(E_Q[c] + \widehat{\mathcal{D}}(Q) \right).$$

What is the Primal Problem associated to EMOT ?

The primal problem is the **nonlinear subhedging value** of c given by

$$\mathfrak{P}(c) = \sup \left\{ \sum_{t=0}^{T} U_{\widehat{Q}_t}(\varphi_t) \mid \varphi \text{ s.t. } \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi_t + I^{\Delta} \leqslant c \right\},$$

where the inequality is required to hold for all $x \in \Omega$.

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where the inequality is required to hold for all $x \in \Omega$.
The terms $E_{\widehat{Q}_t}[\varphi_t]$ in the classical MOT are here replaced by the nonlinear evaluations:

$$U_{\widehat{Q}_t}(\varphi_t) = \sup_{\alpha,\lambda \in \mathbb{R}} \left(\int_{K_t} u_t(\varphi_t(x_t) + \alpha \mathrm{Id}_t(x_t) + \lambda) \mathrm{d}\widehat{Q}_t(x_t) - (\alpha x_0 + \lambda) \right),$$

associated to the utility u_t in $\mathcal{D}_{v_t^*, \widehat{Q}_t}(Q_t)$, via $v_t^* = -u_t^*$.
The functional $U_{\widehat{Q}_t}(\varphi_t)$ is the **cash additive and stock additive** version of the expected utility functional

$$\varphi_t \to \int_{K_t} u_t(\varphi_t(x_t)) \mathrm{d}\widehat{Q}_t(x_t)$$

Nonlinear Pricing-Hedging Duality

As a consequence of our main results we prove the following duality:

$$\inf_{Q\in\mathrm{Mart}(\Omega)}\left(E_Q\left[c\right]+\sum_{t=0}^{T}\mathcal{D}_{v_t^*,\widehat{Q}_t}(Q_t)\right)$$

$$=\sup\left\{\sum_{t=0}^{T}U_{\widehat{Q}_t}(\varphi_t)\mid\varphi\text{ s.t. }\exists\Delta\in\mathcal{H}\text{ s.t. }\sum_{t=0}^{T}\varphi_t(x_t)+I^{\Delta}(x)\leqslant c(x)\right\}$$

$$=\sup_{\Delta\in\mathcal{H}}\sup\left\{\sum_{t=0}^{T}U_{\widehat{Q}_t}(\varphi_t)\mid\varphi\text{ s.t. }\sum_{t=0}^{T}\varphi_t(x_t)+I^{\Delta}(x)\leqslant c(x)\right\}.$$

Heuristic approach

$$\inf_{Q\in \mathrm{Mart}(\Omega)} \left(E_Q\left[c\right] + \mathcal{D}(Q)\right)$$

$$= \inf_{Q\in \mathrm{Prob}(\Omega)} \sup_{\Delta\in\mathcal{H}} \left(E_Q\left[c - \sum_{t=0}^{T-1} \Delta_t(X_0,\dots,X_t)(X_{t+1}-X_t)\right] + \mathcal{D}(Q)\right)$$

$$= \inf_{\mu\in \mathrm{Meas}(\Omega)} \sup_{\substack{\Delta\in\mathcal{H}\\ \xi\in\mathbb{R}^{T+1}}} \left(\int_\Omega \left[c - I^\Delta + \sum_{t=0}^{T}\xi_t\right] \mathrm{d}\mu - \sum_{t=0}^{T}\xi_t + \mathcal{D}(\mu)\right)$$

$$= \inf_{\mu\in \mathrm{Meas}(\Omega)} \sup_{\substack{\Delta\in\mathcal{H}\\ \xi\in\mathbb{R}^{T+1}}} \left(\int_\Omega \left[c - I^\Delta + \sum_{t=0}^{T}\xi_t\right] \mathrm{d}\mu - \sum_{t=0}^{T}\xi_t + \right.$$

$$\left. + \sup_{\varphi\in\mathcal{E}} \left(U(\varphi) - \int_\Omega \left(\sum_{t=0}^{T}\varphi_t\right)\mathrm{d}\mu\right)\right)$$

...Heuristic continues

$$= \sup_{\substack{\Delta\in\mathcal{H},\varphi\in\mathcal{E},\\ \xi\in\mathbb{R}^{T+1}}} \left(\inf_{\mu\in\mathrm{Meas}(\Omega)} \int_\Omega \left[c - I^\Delta + \sum_{t=0}^T \xi_t - \sum_{t=0}^T \varphi_t \right] \mathrm{d}\mu - \sum_{t=0}^T \xi_t + U(\varphi) \right)$$

$$= \sup_{\substack{\Delta\in\mathcal{H},\\ \xi\in\mathbb{R}^{T+1}}} \sup \left\{ U(\varphi) - \sum_{t=0}^T \xi_t \mid \varphi \in \mathcal{E}, c - I^\Delta + \sum_{t=0}^T \xi_t - \sum_{t=0}^T \varphi_t \geqslant 0 \right\}$$

$$= \sup_{\substack{\Delta\in\mathcal{H},\\ \xi\in\mathbb{R}^{T+1}}} \sup \left\{ U(\varphi + \xi) - \sum_{t=0}^T \xi_t \mid \varphi \in \mathcal{E}, \sum_{t=0}^T \varphi_t + I^\Delta \leqslant c \right\}$$

$$= \sup_{\Delta\in\mathcal{H}} \sup \left\{ \sup_{\xi\in\mathbb{R}^{T+1}} \left(U(\varphi + \xi) - \sum_{t=0}^T \xi_t \right) \mid \varphi \in \mathcal{E}, \sum_{t=0}^T \varphi_t + I^\Delta \leqslant c \right\}$$

$$= \sup_{\Delta\in\mathcal{H}} \sup \left\{ S^U(\varphi) \mid \varphi \in \mathcal{E}, \sum_{t=0}^T \varphi_t + I^\Delta \leqslant c \right\}$$

The Optimized Certainty Equivalent

For a given proper and concave functional $U : \mathcal{E} \to [-\infty, +\infty)$ we define the Optimized Certainty Equivalent $S^U : \mathcal{E} \to [-\infty, +\infty]$ by

$$S^U(\varphi) := \sup_{\xi \in \mathbb{R}^{T+1}} \left(U(\varphi + \xi) - \sum_{t=0}^{T} \xi_t \right), \; \mathrm{dom}(S^U) = \{\varphi \in \mathcal{E} \mid S^U(\varphi) > -\infty\}$$

which is Cash Additive and a concave functional on $\mathrm{dom}(S^U)$.

Mathematical Formulation and Main Results

- The main Theorem relies on:
  - a Fenchel-Moreau argument applied to the dual system $(C_{0:T}, (C_{0:T})^*)$, where $C_{0:T}$ is a set of appropriately weighted continuous functions;
  - Daniell-Stone Theorem that guarantees that the elements, in the dual space $(C_{0:T})^*$, that enter in the dual representation can be represented by probability measures.
- In order to make this possible, an order type continuity assumption on the valuation functional U will be required.

Key features

1. General penalization terms $\mathcal{D}$
2. Non linear subhedging functional: Optimized Certainty Equivalent S^U
3. The claim c is lsc and superlinear, $K_0, \ldots, K_T$ are closed subsets of $\mathbb{R}$
4. General cone $\mathcal{A}$ of dynamic hedging instruments (not only stochastic integrals), so that in the duality we obtain not only martingale measure, but possibly ε-martingale / super / sub martingale measures...
5. Possibility to choose for static hedging also path dependent options (not only those with a given maturity t)
6. Multiperiod discrete time and multidimensional stock process (but for this presentation we take only one dimensional stock X).

Setup: the spaces of continuous weighted functions

For a metric space $\mathbb{X}$, we introduce the following notations

- $\mathcal{C}(\mathbb{X})$ is the class of real-valued, continuous functions on $\mathbb{X}$
- For a $\psi \in \mathcal{C}(\mathbb{X})$ we set

$$C_\psi := \left\{ \phi \in \mathcal{C}(\mathbb{X}) \mid \|\phi\|_\psi := \sup_{x \in \mathbb{X}} \frac{|\phi(x)|}{1 + |\psi(x)|} < +\infty \right\} .$$

- C_ψ is a Banach lattice under the norm $\|\cdot\|_\Psi$.
- The topological dual of C_ψ will be denoted by $(C_\psi)^*$.
- Every $\phi \in C_\psi$ satisfies: $|\phi(x)| \leqslant \|\phi\|_\psi \left(1 + \sum_{t=0}^T |\psi(x)|\right)$
- $m\mathcal{B}(\mathbb{X})$ is the class of real-valued, Borel-measurable functions on $\mathbb{X}$

Setup: Ω, $C_{0:T}$

Fix $T+1$ closed (not necessarily compact) subsets of $\mathbb{R}$: $K_0, \ldots, K_T$.

$$\Omega_t = K_0 \times \cdots \times K_t, \quad \Omega = K_0 \times \cdots \times K_T.$$

$$B_{0:T} := \left\{ \phi \in m\mathcal{B}\left(\Omega\right) \mid \|\phi\|_{0:T} := \sup_{x \in \Omega} \frac{|\phi(x)|}{1 + \sum_{t=0}^{T} |x_t|} < +\infty \right\}$$

$$C_{0:t} := \left\{ \phi \in \mathcal{C}\left(\Omega_t\right) \mid \|\phi\|_{0:t} := \sup_{x \in \Omega_t} \frac{|\phi(x_0, \ldots, x_t)|}{1 + \sum_{s=0}^{t} |x_s|} < +\infty \right\}$$

$$C_t := \left\{ \phi \in \mathcal{C}\left(K_t\right) \mid \|\phi\|_t := \sup_{x_t \in K_t} \frac{|\phi(x_t)|}{1 + |x_t|} < +\infty \right\}$$

$X_0, \ldots, X_T$ the canonical projections $X_t : \Omega \to K_t$, for $t = 0, \ldots, T$.

Setup: The space $\mathcal{E}$ for static hedging

- Fix vector subspaces $\mathcal{E}_0, \ldots, \mathcal{E}_T$ with

$$\mathbb{R} \subseteq \mathcal{E}_t \subseteq C_{0:t}, \;\; t = 0, 1, \ldots, T.$$

- The space $\mathcal{E}_t$ is formed by the elements (continuous weighted functions) that we may use for static hedging (i.e. cash, stocks, options with maturity t, or more complex options)
- Observe that when $\varphi \in \mathcal{E}_t \subseteq C_{0:t}$, then $\varphi = \varphi(x_0, \ldots, x_t)$,
- In the additive structure, to be introduced later, we will require $\varphi \in \mathcal{E}_t \subseteq C_t$, so that $\varphi = \varphi(x_t)$.
- We fix:

$$\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T \, .$$

Setup: the spaces of dual elements $\mathrm{Meas}^1(\Omega))$ and $\mathrm{Prob}^1(\Omega))$

Every finite signed measure γ on the Borel σ-algebra $\mathcal{B}(\Omega)$ such that

$$C_{0:T} \subseteq L^1(\Omega, \mathcal{B}(\Omega), |\gamma|)$$

induces a continuous linear functional $\eta \in (C_{0:T})^*$ via integration:

$$c \mapsto \langle c, \eta \rangle = \int_\Omega c \, \mathrm{d}\gamma \,, \;\; \forall \, c \in \; C_{0:T} \,.$$

The collection of such functionals, identified with the corresponding measures, will be denoted by $\mathrm{ca}^1(\Omega)$, while the class of non negative measures (resp. probability measures) in $\mathrm{ca}^1(\Omega)$ will be denoted by

$$\mathrm{Meas}^1(\Omega)) \,, \;\; \mathrm{Prob}^1(\Omega)).$$

Setup: the valuation functional U and its conjugate $\mathcal{D}$

- Let $U : \mathcal{E} \to [-\infty, +\infty)$ be a proper, concave functional and consider the proper, convex functional.

$$V(\varphi) := -U(-\varphi)\,,$$

- We define $\mathcal{D} : (C_{0:0})^* \times (C_{0:1})^* \times \cdots \times (C_{0:T})^* \to (-\infty, +\infty]$ by

$$\mathcal{D}(\gamma_0, \ldots, \gamma_T) := \sup_{\varphi \in \mathcal{E}} \left(U(\varphi) - \sum_{t=0}^{T} \langle \varphi_t, \gamma_t \rangle \right), \quad \gamma \in \bigtimes_{t=0}^{T} (C_{0:t})^*.$$

- For $\gamma \in (C_{0:T})^*$ we define $\mathcal{D}(\gamma)$ using its restrictions.

The subhedging value

Definition

Given a convex cone $\mathcal{A} \subseteq C_{0:T}$ and a measurable $c \in m\mathcal{B}(\Omega)$ we define

$$\mathfrak{P}(c) := \sup_{z \in -\mathcal{A}} \sup_{\varphi \in \mathbf{\Phi}_z(c)} S^U(\varphi) \in [-\infty, +\infty]$$

$$\mathbf{\Phi}_z(c) := \left\{ \varphi \in \mathrm{dom}(S^U) \mid \sum_{t=0}^{T} \varphi_t(x_0, \ldots, x_t) + z(x) \leqslant c(x) \ \ \forall\, x \in \Omega \right\},$$

and the usual convention $\sup \varnothing = -\infty$ is adopted.

On the set of dynamic hedging instruments $\mathcal{A}$

- A general convex cone $\mathcal{A}$ will replace the set of terminal values of stochastic integrals (as the following examples will show).
- Let $\mathcal{A}^\circ$ be the polar cone of $\mathcal{A}$

$$\mathcal{A}^\circ := \{\eta \in (C_{0:T})^* \mid \langle z, \eta \rangle \leqslant 0 \ \forall z \in \mathcal{A}\}$$

  and we observe that for any $\eta \in (C_{0:T})^*$

$$\sigma_{\mathcal{A}}(\eta) := \sup_{z \in \mathcal{A}} \langle z, \eta \rangle = \begin{cases} 0 & \eta \in \mathcal{A}^\circ \\ +\infty & \text{otherwise} \end{cases} .$$

- The set $\mathrm{Prob}^1(\Omega) \cap \mathcal{A}^\circ$ will appear in the main duality result.

$\mathcal{A}$:={stochastic integrals}

Example

Martingale measures. Set

$$\mathcal{H} := \{\Delta = [\Delta_0, \dots, \Delta_{T-1}] \mid \Delta_t \in \mathcal{C}_b(K_0 \times \dots \times K_t)\}$$

$$I^\Delta(x) := \sum_{t=0}^{T-1} \Delta_t(x_0, \dots, x_t)(x_{t+1} - x_t) \;\; \forall\, x \in \Omega$$

$$\mathcal{A} = \mathcal{I} := \left\{ I^\Delta \mid \Delta \in \mathcal{H} \right\} \subseteq C_{0:T}\,.$$

$$\mathrm{Mart}(\Omega) := \{Q \in \mathrm{Prob}^1(\Omega) \mid E_Q[I^\Delta] = 0,\ \forall \Delta \in \mathcal{H}\} = \mathrm{Prob}^1(\Omega) \cap \mathcal{A}^\circ.$$

Other possible choices for $\mathcal{A}$

Example

(ε-martingale measures) For every $\varepsilon \geqslant 0$, the set of ε-martingale measures (see Guo Obloj 2019) is

$$\mathrm{Mart}_\varepsilon(\Omega) := \left\{ Q \in \mathrm{Prob}^1(\Omega) \mid E_Q\left[I^\Delta\right] \leqslant \varepsilon \sum_{t=0}^{T-1} \|\Delta_t\|_\infty \,,\ \forall \Delta \in \mathcal{H} \right\} .$$

Thus, taking

$$\mathcal{A}^\varepsilon := \mathrm{convex}\left(\left\{ I^\Delta - \varepsilon \sum_{t=0}^{T-1} \|\Delta_t\|_\infty \,, \Delta \in \mathcal{H} \right\} \right) \subseteq C_{0,T}$$

(here $\mathrm{convex}(\cdot)$ stands for the convex hull in $C_{0:T}$, which is easily seen to be a cone since $\mathcal{H}$ is a vector space), one sees that

$$\mathrm{Mart}_\varepsilon(\Omega) = \mathrm{Prob}^1(\Omega) \cap (\mathcal{A}^\varepsilon)^\circ .$$

Other possible choices for $\mathcal{A}$

Example

(Super/submartingale measures) Alternative choices for the set $\mathcal{A}$ are $\mathcal{A}^{\pm} = \{I^{\Delta} \mid \Delta \in (\mathcal{H}^{\pm})\}$ where $\mathcal{H}^{+} = \{\Delta \in \mathcal{H} \mid \Delta_t \geqslant 0 \ \forall t = 0, \dots, T\}$ and $\mathcal{H}^{-} = -\mathcal{H}^{+}$. The choice $\mathcal{A}^{+}$ yields

$$\{\text{super martingale measures for canonical process}\} = \mathrm{Prob}^1(\Omega) \cap (\mathcal{A}^{+})^{\circ}$$

and $\mathcal{A}^{+}$ models dynamic trading with no short selling.

Example

For any set $\mathcal{A}$ such that $\{0\} \subseteq \mathcal{A} \subseteq -(C_{0:T})_{+}$ we obtain:

$$\mathrm{Prob}^1(\Omega) = \mathrm{Prob}^1(\Omega) \cap \mathcal{A}^{\circ}.$$

This choice leads to the Entropy Optimal Transport duality with no dynamic hedging (no martingales).

The assumptions

(i) (Structural Assumption): Let $K_0, \ldots, K_T$ be closed subsets of $\mathbb{R}$. The functional $U : \mathcal{E} \to [-\infty, +\infty)$ is concave with $U(0) \in \mathbb{R}$ and $V(\varphi) := -U(-\varphi)$. $\mathcal{A} \subseteq C_{0:T}$ is a convex cone with $0 \in \mathcal{A}$.

The assumptions

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(ii) (Continuity type Assumption): For every $t = 0, \dots, T$ there exist a sequence of compact sets $I_t(n) \subseteq K_t$, $n \geqslant 1$ and a sequence of functions $0 \leqslant f_t^n \in \mathcal{E}_t$, $n \geqslant 1$ s.t.:

$$1 + \sum_{t=0}^{T} |x_t| \leqslant \sum_{t=0}^{T} f_t^n(x_0, \dots, x_t) \;\; \forall [x_0, \dots, x_T] \in \Omega \backslash I_0(n) \times \cdots \times I_T(n)$$

and

$$V(\beta f_0^n, \dots, \beta f_T^n) \to_n 0 \quad \forall\, \beta \in \mathbb{R}, \beta > 0\,.$$

On the Assumptions

- In the particular case $K_0, \ldots, K_T \subseteq [0, +\infty)$, Assumption (ii) holds if: there exists a sequence of call options $f_t^n \in \mathcal{E}_t$ such that valuation V over this sequence converge to zero when the corresponding strikes diverge to infinity.
- Such Assumption (ii) is inspired by Cheridito, Kupper, Tangpi (2017).

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- Assumption (ii) (and Daniell-Stone Theorem) guarantees that the infimum in the dual representation is taken w.r. to elements in the dual space $(C_{0:T})^*$ that can be represented by probability measures.

The main result

Theorem

(i) *If*

$$\mathfrak{P}(\hat{c}) < +\infty \text{ for some } \hat{c} \in B_{0:T},$$

then $\mathfrak{P}(c) \in \mathbb{R}$ *for every* $c \in B_{0:T}$ *and* $\mathfrak{P} : B_{0:T} \to \mathbb{R}$ *is norm continuous, cash additive, concave and nondecreasing on* $B_{0:T}$*;*

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(ii) *For every lower semicontinuous* $c : \Omega \to (-\infty, +\infty]$ *satisfying*

$$c(x) \geqslant -A\left(1 + \sum_{t=0}^{T} |x_t|\right) \quad \forall x \in \Omega, \quad \text{for some } A \in [0, +\infty), \tag{3}$$

we have

$$\mathfrak{P}(c) := \sup_{z \in -\mathcal{A}} \sup_{\varphi \in \mathbf{\Phi}_z(c)} S^U(\varphi) = \inf_{Q \in \mathrm{Prob}^1(\Omega) \cap \mathcal{A}^\circ} \left(E_Q[c] + \mathcal{D}(Q)\right) \tag{4}$$

Furthermore, if $\mathfrak{P}(c) < +\infty$ *the infimum in* (4) *is a minimum.*

Remarks

- The condition $\mathfrak{P}(\hat{c}) < +\infty$ for some $\hat{c} \in B_{0:T}$ is not required for the validity of the Theorem Item (ii).
- We allow $\mathrm{Prob}^1(\Omega) \cap \mathcal{A}^\circ = \varnothing$, with the usual convention $\inf \varnothing = +\infty$.
- Moreover,

$$\text{There exists } Q \in \mathrm{Prob}^1(\Omega) \cap \mathcal{A}^\circ \text{ s.t. } \mathcal{D}(Q) < +\infty \Leftrightarrow \mathfrak{P}(0) < +\infty.$$

Convergence of EMOT problems

For each $n \in \mathbb{N} \cup \{\infty\}$ consider: a functional U_n, a set $\mathcal{A}_n \subseteq C_{0:T}$ and the subhedging problem

$$\mathfrak{P}_n(c) := \sup_{z \in -\mathcal{A}_n} \sup_{\varphi \in \mathbf{\Phi}_z(c)} S^{U_n}(\varphi)$$

Proposition

Suppose that, for each $n \in \mathbb{N} \cup \{\infty\}$, the same assumptions of the Theorem hold for $\mathfrak{P}_n(c)$ and that $\mathfrak{P}_n(c) < +\infty$. Suppose that

$$\mathcal{D}_\infty(Q) + \sigma_{\mathcal{A}_\infty}(Q) = \sup_{n \in \mathbb{N}} \left(\mathcal{D}_n(Q) + \sigma_{\mathcal{A}_n}(Q)\right)$$

$$\mathcal{D}_{n+1}(Q) + \sigma_{\mathcal{A}_{n+1}}(Q) \geqslant \mathcal{D}_n(Q) + \sigma_{\mathcal{A}_n}(Q), \quad n \in \mathbb{N},$$

for every $Q \in \mathrm{Prob}^1(\Omega)$. Then $\mathfrak{P}_n(c) \uparrow_n \mathfrak{P}_\infty(c)$ for every $c : \Omega \to (-\infty, +\infty]$ which is lower semicontinuous and satisfies the superlinear growth condition (3).

The case of stochastic integral

Given functions $c : \Omega \to (-\infty, +\infty]$, $g : \Omega \to [-\infty, +\infty)$ let

$$\mathcal{S}_{sub}(c) = \left\{ \varphi \in \mathrm{dom}(S^U) \mid \exists \Delta \in \mathcal{H} \text{ s.t. } \sum_{t=0}^{T} \varphi(x_0, \dots, x_t) + I^{\Delta}(x) \leqslant c(x) \ \ \forall\, x \right\}$$

Corollary

Suppose that the assumptions in the Theorem are satisfied, that $g : \Omega \to [-\infty, +\infty)$ is upper semicontinuous and that also the superlinear condition (3) holds replacing c with $-g$. Then

$$\inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[c] + \mathcal{D}(Q) \right) = \sup_{\varphi \in \mathcal{S}_{sub}(c)} S^U(\varphi) \ ,$$

$$\sup_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[g] - \mathcal{D}(Q) \right) = \inf_{\varphi \in \mathcal{S}_{sup}(g)} S_V(\varphi) \ .$$

where $\mathcal{S}_{sup}(g) := -\mathcal{S}_{sub}(-g)$ and $S_V(\varphi) := -S^U(-\varphi)$. If the LHS are finite, there exists an optimum in the LHS.

Towards the applications

- We now will consider applications where we will adopt the additive structure:

$$U(\varphi_0, \ldots, \varphi_T) = \sum_{t=0}^{T} U_t(\varphi_t), \quad \varphi_t = \varphi_t(x_t).$$

- Alternatively, we will start by defining the penalization term $\mathcal{D}_t$, and then build

$$\mathcal{D}(Q) := \sum_{t=0}^{T} \mathcal{D}_t(Q_t)$$

and the associated dual functional U.

Additive structure

Proposition

Suppose that $\mathcal{E}_t \subseteq C_t$*, that* $S_t : \mathcal{E}_t \to \mathbb{R}$ *is a concave, cash additive functional null in* 0 *and let* $U(\varphi) := \sum_{t=0}^{T} S_t(\varphi_t)$*. Suppose that the Assumptions (i) and (ii) are fulfilled. Consider* $\forall t = 0, \ldots, T$ *the penalizations*

$$\mathcal{D}_t(Q_t) := \sup_{\varphi_t \in \mathcal{E}_t} \left(S_t(\varphi_t) - \int_{K_t} \varphi_t \, \mathrm{d}Q_t \right) \quad \text{for } Q_t \in \mathrm{Prob}^1(K_t).$$

Let $c : \Omega \to (-\infty, +\infty]$ *be lower semicontinuous and s.t.* (3) *holds. Then*

$$\sup \left\{ \sum_{t=0}^{T} S_t(\varphi_t) \mid \exists \Delta \in \mathcal{H} \text{ with } \sum_{t=0}^{T} \varphi_t(x_t) + I^{\Delta}(x) \leqslant c(x) \ \forall x \in \Omega \right\}$$

$$= \inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[c] + \sum_{t=0}^{T} \mathcal{D}_t(Q_t) \right)$$

Assumptions on the utility functions and divergences

From now on we suppose that:

- There exists a probability $\widehat{Q} \in \mathrm{Mart}(\Omega)$, having marginals $\widehat{Q}_0, \dots, \widehat{Q}_T$.
- $u_0, \dots, u_T : \mathbb{R} \to [-\infty, +\infty)$ are concave, upper semicontinuous nondecreasing utility functions with $u_0(0) = \dots = u_T(0) = 0$, $u_t(x) \leqslant x \ \forall \, x \in \mathbb{R}$
- For each $t = 0, \dots, T$ we define $v_t(x) := -u_t(-x)$, $x \in \mathbb{R}$ and

$$v_t^*(y) := \sup_{x \in \mathbb{R}} (xy - v_t(x)) = \sup_{x \in \mathbb{R}} (u_t(x) - xy)), \quad y \in \mathbb{R} \, .$$

- We introduce the induced divergences for $Q \in \mathrm{Prob}^1(\Omega)$

$$\mathcal{D}_{\widehat{Q}}(Q) := \begin{cases} \int_\Omega v^* \left(\frac{\mathrm{d}Q}{\mathrm{d}\widehat{Q}} \right) \mathrm{d}\widehat{Q} & \text{if } Q \ll \widehat{Q} \\ +\infty & \text{otherwise} \end{cases} .$$

Hedging only with cash, stocks and options

- As in Beiglböck et al. 2013, we now suppose that the elements in $\mathcal{E}_t \subseteq C_t$ represent portfolios of call options, stocks (i.e.: x_t) and cash, that is $\mathcal{E}_t$ consists of all the functions with the following form:

$$\varphi_t(x_t) = a + bx_t + \sum_{n=1}^{N} c_n(x_t - K_n)^+, \text{ for } a, b, c_n, K_n \in \mathbb{R},\ x_t \in K_t \tag{5}$$

  and take $\mathcal{E} = \mathcal{E}_0 \times \cdots \times \mathcal{E}_T$.
- Let $U_{\widehat{Q}_t}(\varphi_t)$ be the valuation functional associated to u_t:

$$U_{\widehat{Q}_t}(\varphi_t) := \sup_{\alpha,\lambda\in\mathbb{R}} \left(\int_{K_t} u_t(\varphi_t(x_t) + \alpha \mathrm{Id}_t(x_t) + \lambda) \mathrm{d}\widehat{Q}_t(x_t) - (\alpha x_0 + \lambda) \right),$$

- Let $\widehat{Q} \in \mathrm{Mart}(\Omega)$ such that

$$\int_{K_t} v_t\left(\alpha(1 + |x_t|)\right) \mathrm{d}\widehat{Q}_t(x_t) < +\infty \quad \forall \alpha > 0, t = 0, \ldots, T.$$

The duality using cash, stocks and options

Corollary

For each $t = 0, \dots, T$ *suppose* $\mathrm{dom}(u_t) = \mathbb{R}$ *and take the vector space* $\mathcal{E}_t$ *of functions in the form* (5)*. Suppose that* $c : \Omega \to (-\infty, +\infty]$ *is lower semicontinuous and with superlinear growth. Then*

$$\sup\left\{\sum_{t=0}^{T} U_{\widehat{Q}_t}(\varphi_t) \mid \exists \Delta \in \mathcal{H} \text{ with } \sum_{t=0}^{T} \varphi_t(x_t) + I^{\Delta}(x) \leqslant c(x)\ \forall x \in \Omega\right\}$$
$$= \inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[c] + \sum_{t=0}^{T} \mathcal{D}_{v_t^*, \widehat{Q}_t}(Q_t) \right)$$

- With the particular choice $u_t(x) = x$, $t = 0, \dots, T$ we will recover the classical MOT case. Indeed $v_t^* = \delta_{\{1\}}$ and $\mathcal{D}_{v_t^*, \widehat{Q}_t} = \delta_{\widehat{Q}_t}$.
- In case $\mathrm{dom}(u_t) \subset \mathbb{R}$ the duality will hold but with a singular component in the divergence.

Two applications where we start from the penalty $\mathcal{D}_t$

- Penalization terms $\mathcal{D}_t$ induced by market data
- Penalization terms $\mathcal{D}_t$ induced by Wassertein distance

In such applications we will apply the convergence result $\mathfrak{P}_n(c) \uparrow_n \mathfrak{P}_\infty(c)$ already stated.

Loss functions

To construct the penalization terms $\mathcal{D}_t$ we use the following

Definition

A function $G : \mathbb{R} \to (-\infty, +\infty]$ is a loss function if it is convex, nondecreasing, lower semicontinuous and such that $G(0) = 0$. Its conjugate $G^* : \mathbb{R} \to (-\infty, +\infty]$ is $G^*(y) = \sup_{x\in\mathbb{R}}(xy - G(x))$ and satifies $G^*(y) = +\infty$ for every $y < 0$.

Examples:

- power-like loss functions, i.e. $G(x) = \frac{x^p}{p}$ for $x > 0$ ($p > 1$), and $G(x) = 0$ for $x \leqslant 0$.
- A threshold loss function

$$G(x) = \begin{cases} 0 & \text{if } x \leqslant \varepsilon \\ +\infty & \text{otherwise} \end{cases} \implies G_t^*(y) = \varepsilon y \quad \text{for } y \geqslant 0\,.$$

Assumptions for the next two applications

- $\Omega := K_0 \times \cdots \times K_T$ for compact sets $K_0, \ldots, K_T \subseteq \mathbb{R}$;
- $K_0 = \{x_0\}$ for some $x_0 \in \mathbb{R}$;
- $c : \Omega \to (-\infty, +\infty]$ is lower semicontinuous
- $\widehat{Q} \in \mathrm{Mart}(\Omega)$ is a given martingale measure with marginals $\widehat{Q}_0, \ldots, \widehat{Q}_T$ and such that $c \in L^1(\widehat{Q})$.

Penalization with market prices

- For each $t = 0, \dots, T$, consider the functions $(f_{t,n})_{1 \leqslant n \leqslant N_t} \subseteq \mathcal{C}_b(K_t)$ that represent **payoffs of options whose prices** $(c_{t,n})_{1 \leqslant n \leqslant N_t} \subseteq \mathbb{R}$ **are known from the market**.

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- Consider loss functions $G_{n,t} : \mathbb{R} \to (-\infty, +\infty]$.
- Suppose that our measure $\widehat{Q} \in \mathrm{Mart}(\Omega)$ satisfies the proximity condition

$$\left| \int_{K_t} f_{t,n} \, \mathrm{d}\widehat{Q}_t - c_{t,n} \right| \in \mathrm{dom}(G_{t,n}) \quad \forall t = 0, \ldots, T, n = 0, \ldots, N_t$$

- Recall that the robust subhedging price is

$$\inf_{Q \in \mathrm{Mart}(\Omega)} E_Q[c] = \sup \left\{ m \in \mathbb{R} \mid \exists \Delta \in \mathcal{H} \text{ with } m + I^\Delta \leqslant c \right\} := \Pi^{\mathrm{sub}}(c) \, .$$

The duality when penalization with market prices

Proposition

For the penalization terms

$$\mathcal{D}_t^G(Q_t) := \sum_{n=1}^{N_t} G_{t,n}\left(\left|\int_{K_t} f_{t,n}\,\mathrm{d}Q_t - c_{t,n}\right|\right) \quad \textit{for } Q_t \in \mathrm{Mart}(K_t)$$

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we have

$$\inf_{Q\in\mathrm{Mart}(\Omega)}\left(E_Q[c] + \sum_{t=0}^{T}\mathcal{D}_t^G(Q_t)\right) = \sup\left\{\sum_{t=0}^{T} U_t^G(\varphi_t) \mid \varphi \in \mathcal{S}_{sub}(c)\right\},$$

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where

$$U_t^G(\varphi_t) := \sup_{y_t\in\mathbb{R}^{N_t}}\left(\Pi^{sub}\left(\varphi_t + \sum_{n=1}^{N_t} y_{t,n}(f_{t,n} - c_{t,n})\right) - \sum_{n=1}^{N_t} G_{t,n}^*(|y_{t,n}|)\right).$$

Convergence with increasing market information

- The information on the marginal distributions increases, by increasing the number of prices available from the market.
- Let $f_{t,n}(x_t) = (x_t - \alpha_n)^+$, be call options with strikes $(\alpha_n)_n$, and let $(\alpha_n)_n$ be a dense subset of $\mathbb{R}$.
- We take the loss functions $G_{t,n}(x) = 0$ for $x \leqslant 0$ and $G_{t,n}(x) = +\infty$ for all $x > 0$ $t = 0, \ldots, T$, $n \geqslant 1$.

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- For each $k \in \mathbb{N}$ let $c_{t,1}, \dots, c_{t,N_t(k)}$ be the initial segment of market prices, for sequences $N_t(k) \uparrow_k \infty$ and assume they are all computed under the same martingale measure $\hat{Q} \in \mathrm{Mart}(\Omega)$ and set:

$$\mathcal{D}_k^G(Q) := \sum_{t=0}^{T} \sum_{n=1}^{N_t(k)} G_{t,n} \left(\left| \int_{K_t} f_{t,n} \, \mathrm{d}Q_t - c_{t,n} \right| \right)$$

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then:

$$\inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[c] + \mathcal{D}_k^G(Q) \right) \uparrow_k \inf_{Q \in \mathcal{M}(\widehat{Q}_0, \widehat{Q}_1, \dots \widehat{Q}_T)} E_Q[c]$$

Wasserstein-induced penalization

- Call $W_t : \mathrm{Prob}(K_t) \times \mathrm{Prob}(K_t) \to \mathbb{R}$ the (1-)Wasserstein distance.
- Consider loss functions $G_t : \mathbb{R} \to (-\infty, +\infty]$.
- Define

$$Q_t \in \mathrm{Prob}(K_t) \mapsto \mathcal{D}_t^W(Q_t) := \begin{cases} G_t(W_t(Q_t, \widehat{Q}_t)) & \text{for } Q_t \in \mathrm{Mart}_t(K_t) \\ +\infty & \text{otherwise} \end{cases}$$

$$\mathrm{Mart}_t(K_t) = \{Q_t \in \mathrm{Prob}(K_t) \mid \exists\, Q \in \mathrm{Mart}(\Omega) \text{ with } Q_t \equiv Q_t\}$$

The duality for Wasserstein-induced penalization

Proposition

Suppose that there exists a $Q \in \mathrm{Mart}(\Omega)$ *such that* $G_t(W_t(Q_t, \widehat{Q}_t)) < +\infty$, $t = 0, \ldots, T$. *Take* $\mathcal{D}_t^W$, $t = 0, \ldots, T$ *as just defined. Then*

$$\inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q\left[c\right] + \sum_{t=0}^{T} \mathcal{D}_t^W(Q_t) \right) = \sup \left\{ \sum_{t=0}^{T} U_t^W(\varphi_t) \mid \varphi \in \mathcal{S}_{sub}(c) \right\}, \tag{6}$$

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where

$$U_t^W(\varphi_t) := \sup_{\substack{y \geqslant 0 \\ \ell_t \in \mathrm{Lip}(1, K_t)}} \left(\Pi^{sub}(\varphi_t + y\ell_t) - \int_{K_t} y\ell_t \mathrm{d}\widehat{Q}_t - G_t^*(y) \right)$$

If the LHS of (6) *is finite, a minimum point in* (6) *exists.*

A particular Wasserstein penalization

- Recall that $\mathcal{D}_t^W(Q_t) = G_t(W_t(Q_t, \widehat{Q}_t))$
- Let now: $G_t(x) = 0$ if $x \leqslant \varepsilon_t$, $G_t(x) = +\infty$ otherwise. Then:

$$\inf_{Q \in \mathrm{Mart}(\Omega)} \left(E_Q[c] + \sum_{t=0}^{T} \mathcal{D}_t^W(Q_t) \right) = \inf_{Q \in \mathrm{Mart}(\Omega)} \left\{ E_Q[c] \mid W_t(Q_t, \widehat{Q}_t) \leqslant \varepsilon_t \; \forall t \right\}.$$

- Then the formula in the previous slide assigns the (dual) subhedging problem in this case.
- In addition, calling U_{ε_t} the associated U_t as computed above, we verify that $\lim_{\varepsilon_t \downarrow 0} U_{\varepsilon_t}(\varphi_t) = U_0(\varphi_t) = E_{\widehat{Q}}[\varphi_t]$,
- Then applying the previous Proposition, the duality converge, as $\varepsilon_t \downarrow 0$, to the MOT duality.

Convergence with Wasserstein penalization

In real world situations the marginals $\widehat{Q}_0 \ldots, \widehat{Q}_T$ may be known only approximately, and only a sequence of candidates $(\widehat{Q}_t^n)_n \subseteq \mathrm{Prob}(\Omega)$ may be available, which may not even be martingale measures!

Suppose that:

- $\widehat{Q}_t^n \to_n \widehat{Q}_t^\infty$ in the weak sense for probability measures, or equivalently that $W_t(\widehat{Q}_t^n, \widehat{Q}_t^\infty) \to_n 0$.
- For each $n \in \mathbb{N} \cup \{\infty\}$, G_t^n is a loss function with $\sup_{n\in\mathbb{N}} G_t^n(x) = G_t^\infty(x)\ \forall x \in \mathbb{R}$, and $G_t^\infty(x) = +\infty\ \forall x > 0$.
- $\lim_n G_t^n(W_t(\widehat{Q}_t^n, \widehat{Q}_t^\infty)) = 0$.

$$\mathfrak{P}_n^W(c) = \inf_{Q\in\mathrm{Mart}(\Omega)} \left(E_Q[c] + \sum_{t=0}^{T} G_t^n(W_t(Q_t, \widehat{Q}_t^n)) \right) .$$

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Then:

$$\lim_n \mathfrak{P}_n^W(c) = \mathfrak{P}_\infty^W(c) = \inf_{Q\in\mathrm{Mart}(\widehat{Q}_0^\infty,\ldots,\widehat{Q}_T^\infty)} E_Q\left[c\right]$$

Application: indirect utility function

- By considering $\mathcal{E}_t = \mathcal{C}_{0:t}$ we obtain a dual robust representation of the generalized Optimized Certainty Equivalent associated to the indirect utility function.
- Suppose $u_0 = u_1 = \cdots = u_T = u$ and $\mathrm{dom}(u) = \mathbb{R}$
- Let $U^{\mathcal{H}}_{\widehat{Q}} : C_{0,T} \to \mathbb{R}$ be the associated indirect utility under a martingale measure $\widehat{Q} \in \mathrm{Mart}(\Omega)$:

$$U^{\mathcal{H}}_{\widehat{Q}}(\varphi) := \sup_{\Delta \in \mathcal{H}} \int_{\Omega} u(\varphi + I^{\Delta})\, \mathrm{d}\widehat{Q}\,.$$

Dual representation for generalized OCE

Proposition

Let S be the Optimized Certainty Equivalent associated to $U^{\mathcal{H}}_{\widehat{Q}}$, namely $S(\varphi) := \sup_{\xi\in\mathbb{R}} \left(U^{\mathcal{H}}_{\widehat{Q}}(\varphi+\xi)-\xi\right)$, $\varphi \in \mathcal{C}_{0:T}$ *and suppose*

$$\int_{K_t} v\left(\alpha(1+|x_t|)\right) \mathrm{d}\widehat{Q}_t(x_t) < +\infty \quad \forall\alpha > 0, t = 0, \dots, T \,.$$

Then the dual representation of S holds: for every $c \in \mathcal{C}_{0:T}$

$$S(c) = \inf_{Q\in\mathrm{Mart}(\Omega)} \left(E_Q\left[c\right] + \mathcal{D}_{\widehat{Q}}(Q)\right),$$

where for $Q \in \mathrm{Prob}^1(\Omega)$

$$\mathcal{D}_{\widehat{Q}}(Q) := \begin{cases} \int_\Omega v^*\left(\frac{\mathrm{d}Q}{\mathrm{d}\widehat{Q}}\right) \mathrm{d}\widehat{Q} & \textit{if } Q \ll \widehat{Q} \\ +\infty & \textit{otherwise} \end{cases}.$$

Conclusion

1. A non linear robust pricing-hedging duality with options.
2. A non linear robust pricing-hedging duality with options and singular components in the divergence terms.
3. The linear robust pricing-hedging duality with options as in Beiglbock, Henry-Labordere, Penkner (2013) or Acciaio, Beiglbock, Schachermayer, Penkner (2016).
4. A dual robust representation for the Optimized Certainty Equivalent functional.
5. The linear robust pricing-hedging duality without options (similar to Burzoni, F., Maggis (2017)).
6. A robust pricing-hedging duality with penalization function based on market data (compact case)
7. A robust pricing-hedging duality with penalty terms given via Wasserstein distance (compact case).
8. Stability and convergence of EMOT to MOT

Thank you for your attention!