LARGE-SCALE PRINCIPAL-AGENT PROBLEMS

In continuous-time, with volatility control

Emma Hubert[1]

Bachelier Finance Society – One World Seminar – November 11, 2021

[1]Department of Operations Research and Financial Engineering (ORFE), Princeton University, NJ 08544.

CONTENT

MOTIVATION AND LITERATURE

- The functioning of society is largely based on interactions and incentives between (economic) agents.
- Actual example: public authorities seek to incentivise individuals to limit their contacts because their interactions contribute to the spread of the epidemic.
- Two main questions:

  (i) How to model the behaviour of individuals and their interactions towards the epidemic?

  (ii) How can they be optimally encouraged to remain isolated in order to limit the spread?

- Motivation: answering these types of questions in various situations.
- From a mathematical point of view:

  agent behaviour $\Leftrightarrow$ stochastic control problem;

  interactions $\Leftrightarrow$ Nash equilibrium and mean-field games;

  incentives $\Leftrightarrow$ Stackelberg equilibrium, contract theory.

Noteworthy papers: Holmström and Milgrom [7] (1987), Sannikov [11] (2008).

▶ Analyse interactions between economic agents, in particular with asymmetric information.

The principal (she) initiates a contract for a period $[0, T]$.

The agent (he) accepts or not the contract proposed by the principal.

The principal must suggest an optimal contract: maximises her utility, and that the agent will accept (reservation utility).

Asymmetries of information:

Moral Hazard: the agent's behaviour is not observable by the principal (second-best case).

Adverse Selection: a characteristic of the agent is unknown by the principal (third-best case).

- Model by Holmström and Milgrom [7] (1987):

Output process: Stochastic process X with dynamic, for $t \in [0, T]$:

$$dX_t = \alpha_t dt + \sigma_t dW_t.$$

Effort: the agent controls X through the drift α, in order to maximise the following criteria:

$$\mathbb{E}^{\mathbb{P}^\alpha}\Big[- \exp\Big(- R_A \Big(\xi - \int_0^T c(\alpha_t) dt \Big)\Big)\Big].$$

Moral Hazard: the principal only observes X in continuous-time.

- The contract (terminal payment) ξ can only be indexed on X.
- The **optimal** form of contracts for the agent is (see [7, 11]):

$$\xi = \xi_0 - \int_0^T \mathcal{H}(Z_s) ds + \int_0^T Z_s dX_s + \frac{1}{2} R_A \int_0^T Z_s^2 d\langle X \rangle_s, \tag{1}$$

where

(i) Z is a payment rate chosen by the principal;
(ii) $\mathcal{H}$ is the agent's Hamiltonian.

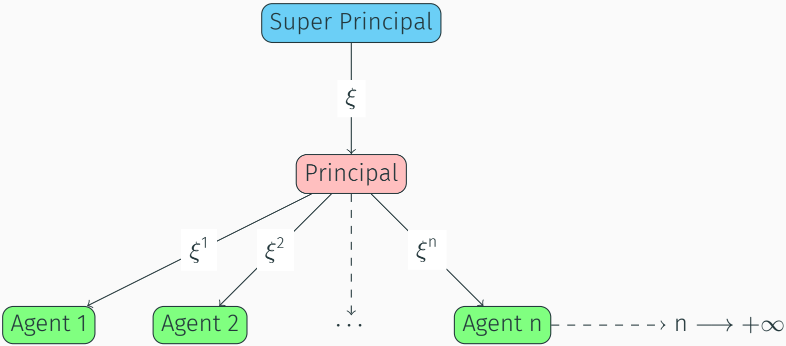

Figure: Extension of the principal-agent model

- Volatility control. Cvitanić, Possamaï, and Touzi [4] (2018)

(i) identify a class of contracts, offered by the principal, that are revealing: the agent's optimal response can be easily calculated;

(ii) prove that this restriction is without loss of generality, using second-order BSDE (2BSDE) ;

(iii) solve the principal's problem, which is now standard.

- Many agents. For example: Élie and Possamaï [5] (2019), and Élie, Mastrolia, and Possamaï [6] (2018) for a continuum of agents.

- Use these recent developments to:

(i) identify the optimal incentives within a hierarchy;

(ii) improve electricity demand management.

INCENTIVES WITHIN A HIERARCHY

- Hierarchy : power entity at the top and subsequent levels of power below.
  - Dominant structure in contemporary society.
  - Raises many questions: efficiency, cost, optimal size... Originally introduced by Knight [9] (1921).
- Incentives within a hierarchy. Link with multi-agents problems, to model information asymmetries : Stiglitz [12] (1975) and Mirrlees [10] (1976).
  - Discrete-time models, usually a single period: Sung [13] (2015).
- Answer two questions:

(i) Interest of continuous-time?

(ii) 'Natural' example where an agent controls the volatility?

Sung [13] (2015) – Pay for performance under hierarchical contracting.

- Hierarchical principal-agent model, with one period and moral hazard.

- Choice of a single-period model: "For ease of exposition and without loss of generality, we formulate a discrete-time model which is analogous to its continuous-time counterpart" (Sung [13] (2015)).

  The principal (she) represents the (risk-neutral) investors of a company.

  The agents are the $n+1$ workers, with CARA utility. Each agent $i \in \{0, \dots, N\}$ produce a random outcome X^i, by carrying out his own task:

  - in Sung's model, $X^i \sim \mathcal{N}(\alpha^i, (\sigma^i)^2)$;
  - in a continuous-time framework, $dX^i_t = \alpha^i_t dt + \sigma^i dW^i_t$, $t \in [0, 1]$.

  The effort of the agent i is represented by α^i, and induces a quadratic cost: $c^i(\alpha^i) = |\alpha^i|^2/2k^i$.

- A manager ($i = 0$) is designated as an intermediary between the principal and the agents.

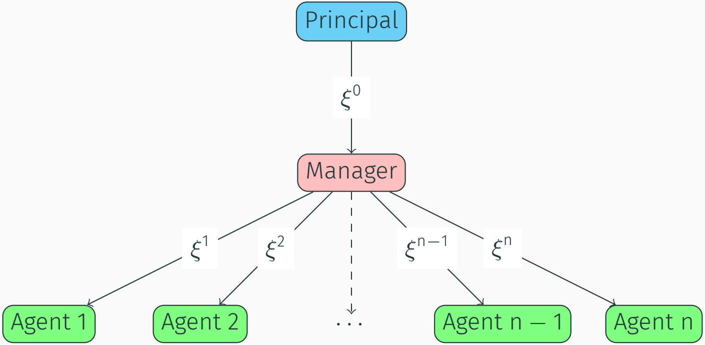

Figure: Sung's model

The i-th agent

- controls the drift of the process X^i with dynamic $dX^i_t = \alpha^i_t dt + \sigma^i dW^i_t$;
- receives a terminal payment ξ^i, function of $(X^i)_{t \in [0,1]}$.

The manager

- controls the drift of a process X^0 with dynamic $dX^0_t = \alpha^0_t dt + \sigma^0 dW^0_t$;
- chooses the contracts ξ^i, by only observing the results X^i.
- receives a terminal payment ξ^0.

The principal observes in continuous-time the process ζ:

$$\zeta_t = \sum_{i=0}^{n} X^i_t - \sum_{i=1}^{n} \xi^i_t, \; t \in [0, 1],$$

and indexes the contract ξ^0 for the manager on this process.

- Interconnected principal-agent problems.

- Given a contract ξ^i, the i-th agent chooses an effort α^i in order to maximise the following utility:

$$\mathbb{E}^{\mathbb{P}^i}\Big[-\exp\Big(-R^i\Big(\xi^i(X^i_\cdot)-\int_0^1 c^i(\alpha^i_t)\mathrm{d}t\Big)\Big)\Big].$$

Assumption: ξ^i can only be indexed on the agent's outcome X^i.

- The optimal form of contracts is (see [7] or [11]):

$$\xi^i=\xi^i_0-\int_0^1\mathcal{H}^i(Z^i_s)\mathrm{d}s+\int_0^1 Z^i_s\mathrm{d}X^i_s+\frac{1}{2}R^i\int_0^1(Z^i_s)^2\mathrm{d}\langle X^i\rangle_s, \tag{2}$$

where Z^i is a process chosen by the manager, and $\mathcal{H}^i(z)=\sup_{a\in\mathbb{R}}\{az-c^i(a)\}$.

- The optimal effort of the agent is $k^iZ^i_t$, and it is possible to compute the dynamics of X^i and ξ^i under this optimal effort.

- Given a contract ξ^0, the manager chooses α^0 and Z^i, for $i = 1, \dots, n$, in order to maximise:

$$\mathbb{E}^{\mathbb{P}^0}\Bigg[-\exp\bigg(-R^0\Big(\xi^0(\zeta_\cdot) - \int_0^1 c^0(\alpha_t^0)\mathrm{d}t\Big)\bigg)\Bigg].$$

Recall: the principal only observes ζ (in continuous-time), which satisfies:

$$\mathrm{d}\zeta_t = \mathrm{d}X_t^0 + \sum_{i=1}^n \bigg(k^i Z_t^i - \frac{1}{2}(Z_t^i)^2\Big(k^i + R^i(\sigma^i)^2\Big)\bigg)\mathrm{d}t + \sigma^i \sum_{i=1}^n (1 - Z_t^i)\mathrm{d}W_t^i.$$

- The manager controls the volatility of his state variable ζ!
- The **optimal** of contract is (Cvitanić, Possamaï, and Touzi [4] (2018)):

$$\xi^0 = \xi_0^0 - \int_0^1 \mathcal{H}^0(Z_s, \Gamma_s)\mathrm{d}s + \int_0^1 Z_s \mathrm{d}\zeta_s + \frac{1}{2}\int_0^1 (\Gamma_s + R^0 Z_s{}^2)\mathrm{d}\langle\zeta\rangle_s. \qquad (3)$$

- Explicit forms for the manager's optimal controls and dynamics for ζ and ξ^0 at the optimum.

- After identifying the form of the revealing contracts, the principal problem is a standard stochastic control problem:

$$V_0 = \sup_{Z,\Gamma} \mathbb{E}^{\mathbb{P}^\star}\left[\zeta_T - \xi^0\right].$$

- Optimal controls are $(Z^\star, \Gamma^\star)$ where $\Gamma^\star := -R^0 (Z^\star)^3$, with $Z^\star$ solution of a maximisation problem.
- Main result: $\Gamma^\star$ is different from the one imposed by Sung, i.e. $\Gamma = -R^0 Z^2$.
- This leads to different results in terms of optimal efforts and value functions!

Why?

- In the one-period model, the principal only observes ζ_1. Since the variance of ζ_1 is controlled (through Z), she cannot observe it, nor can she propose a contract that depends on it.

- Sung limits the study to linear contracts:

$$\xi^0 = \xi_0^0 - \mathcal{H}^0(Z) + Z\zeta + \frac{1}{2}(\Gamma + R^0 Z^2)\mathbb{V}\mathrm{ar}[\zeta] \quad (\Gamma = -R^0 Z^2),$$

and states that this restriction is "without loss of generality, as long as [the] results are interpreted in the context of continuous-time models as in Holmström et Milgrom [7]"

- In continuous time, the principal observes the paths of ζ, and therefore (can estimate) its quadratic variation.

- When the variance is controlled, linear contracts are not optimal (Cvitanić, Possamaï, and Touzi [4] (2018)). In particular, in our framework, $\Gamma^\star := -R^0 Z^3$.

- Shows the need to rigorously study continuous-time, and therefore to use second-order BSDEs.
- Extensions and potential applications:
  - larger hierarchy, more general utility and costs functions, other forms of reporting ζ;
  - work in progress with Bensalem and Hernández-Santibáñez to add jumps in the outputs' dynamics;
  - application to finance (see Keppo, Touzi, and Ruiting [8] (2021)), to insurance (insured – insurer – reinsurer), regulation...
- What about more general contracts, indexed on the results of other workers? What would happen in a mean-field framework?

TOWARDS A MEAN-FIELD OF AGENTS

MOTIVATION: ELECTRICITY DEMAND MANAGEMENT

- Very few electricity storage solutions: **supply-demand equilibrium** at all times ⇒ Acting on the supply side?
- **Problem:** inflexible (or expensive) production and random renewable energies.

- **Idea:** increase the flexibility of the demand, facilitated by the development of smart meters ⇒ Tariff offers, price signals to encourage the consumers to reduce their consumption during peak demand periods.
- **However:** large variance in the consumer's response to these mechanisms.

Aïd, Possamaï, and Touzi [1] (2019) – Principal-agent model with volatility control, to improve the consumer's response.

- Uses the results of Cvitanić, Possamaï, and Touzi [4] (2018).

- Contribution, with Elie, Mastrolia and Possamaï: extension of [1] to a model with a continuum of agents, whose electricity consumption is impacted by a common noise, representing climatic hazards.

Classic MFG framework: all agents are identical.

- Study of a 'normal' consumer, who has no impact on total consumption: the representative agent (he).
- His consumption at time $t \in [0, T]$ is:

$$X_t = x_0 - \int_0^t \alpha_s \cdot \mathbf{1}_d \mathrm{d}s + \int_0^t \sigma(\beta_s) \cdot \mathrm{d}W_s + \int_0^t \sigma^\circ \mathrm{d}W_s^\circ, \tag{4}$$

where

- α, effort to reduce the mean of his consumption;
- β, effort to reduce the volatility ;
- W, d-dim. MB, representing the randomness specific to the agent;
- W°, uni-dim. MB, representing the noise common to all agents.

- Optimisation problem of the representative consumer:

$$V_0^A(\xi) := \sup_{\nu=(\alpha,\beta)} \mathbb{E}^{\mathbb{P}}\left[U_A\left(\xi - \int_0^T (c(\nu_t) - f(X_t))dt\right)\right], \tag{5}$$

where c is the cost of effort, f represents the agent's preference towards his consumption, and $U_A(x) = -e^{-R_A x}$.

- Aïd, Possamaï, and Touzi [1] (2019): Contract indexed on X, and its quadratic variation $\langle X \rangle$, through a process (Z, Γ).
- The principal chooses (Z, Γ) in order to maximise her profit.

- Principal – multi-agents models : the principal can take advantage of the supplementary information available to her (see [5, 6]).

- In our case, the principal can compute the distribution, conditional to common noise, of the consumption of the others, denoted $\hat{\mu}$.

$$\Rightarrow \text{New form of contract: } \xi(X, \hat{\mu}).$$

- Using the 'chain rule with common noise' by Carmona and Delarue [3] (2018), 'revealing contracts' should be of the form:

$$\xi_T = \xi_0 - \int_0^T \mathcal{H}(X_s, \zeta_s, \hat{\alpha}_s^\star, \hat{\mu}_s)ds + \int_0^T Z_s dX_s + \frac{1}{2}\int_0^t \big(\Gamma_s + R_A Z_s^2\big)d\langle X\rangle_s$$
$$+ \int_0^T \widehat{\mathbb{E}}^{\hat{\mu}_s}\Big[Z_s^\mu(\widehat{X}_s)d\widehat{X}_s\Big] + \int_0^T \widetilde{f}\big(\widehat{\mu}_s, Z_s, Z_s^\mu\big)ds,$$

  - $\widehat{\alpha}^\star$, the optimal effort of others on the drift of their consumption,
  - $\widehat{X}$, the consumption of others;
  - $\widehat{\mathbb{E}}^{\hat{\mu}}$, expectation under $\hat{\mu}$ (with respect to the common noise);
  - $\zeta_t = \big(Z_t, \Gamma, Z_t^\mu\big)$, parameters optimised by the principal.

Equilibrium between agents: Given a contract of the previous form, indexed by $\zeta_t = (Z_t, \Gamma, Z^\mu_t)$,

- the optimal effort of the representative agent does not depend on Z^μ or $\hat{\mu}$;
- mean-field equilibrium: the optimal efforts are the same for all consumers, and thus $\widehat{X} \overset{\mathcal{L}}{\sim} X$ and $\hat{\mu} = \mu^X$;

Principal's problem :

- this form of contract, where the principal chooses $\zeta := (Z, \Gamma, Z^\mu)$, is **without loss of generality** $\Leftrightarrow$ second-order BSDE of the mean-field type;
- from the principal's point of view, the contract ξ is a function of X and μ^X, the conditional law of X. $\Leftrightarrow$ Problem of McKean-Vlasov type.

- Let X° be the consumption without common noise (corrected for climatic hazards) :

$$dX_t^\circ = -\alpha^\star(Z_t^\star)dt + \sigma^\star(\Gamma_t^\star) \cdot dW_t.$$

- Rewriting of the contract: indexed on X° and W°:

$$\xi_T = \xi_0 - \int_0^T \mathcal{H}(X_s, \zeta_s^\star)ds + \int_0^T Z_s^\star dX_s^\circ + \frac{1}{2}\int_0^T \left(\Gamma_s^\star + R_A|Z_s^\star|^2\right)d\langle X^\circ\rangle_s$$
$$+ R_P\sigma^\circ \int_0^T \bar{f}(s,\mu^X)dW_s^\circ + \frac{1}{2}R_A R_P^2|\sigma^\circ|^2 \int_0^T |\bar{f}(s,\mu^X)|^2 ds.$$

- Risk-neutral case ($R_P = 0$) $\Rightarrow$ Classic contract for drift and volatility control, indexed on X°, the part of the consumption that is actually controlled by the agent.

CONCLUSION AND “TRENDY” IDEAS

Theoretical contribution: Extension of PA problems with volatility control to a multitude / continuum of agents, by developing natural extensions of the 2BSDE theory.

Applications:

- modelling of interactions and incentives in an organisation;
- demand-response management;
- finance, insurance, regulation;
- control of an epidemic?

Control via social distancing: political or individual choice?

- Viewpoint of a global planner.
- Individual point of view, with R. Elie and G. Turinici.
- Governmental point of view, with T. Mastrolia, D. Possamaï and X. Warin.
- Government dealing with a mean-field of individuals, by Aurell, Carmona, Dayanikli, and Lauriere [2] (2020).

BIBLIOGRAPHY

[1] R. Aïd, D. Possamaï and N. Touzi. Optimal electricity demand response contracting with responsiveness incentives. Preprint arXiv:1810.09063, 2019.

[2] A. Aurell, R. Carmona, G. Dayanikli and M. Lauriere. Optimal incentives to mitigate epidemics: a Stackelberg mean field game approach. Preprint arXiv:2011.03105, 2020.

[3] R. Carmona and F. Delarue. Probabilistic theory of mean field games with applications II, volume 84 of Probability theory and stochastic modelling. Springer International Publishing, 2018.

[4] J. Cvitanić, D. Possamaï and N. Touzi. Dynamic programming approach to principal–agent problems. Finance and Stochastics, 22(1):1–37, 2018.

[5] R. Élie and D. Possamaï. Contracting theory with competitive interacting agents. SIAM Journal on Control and Optimization, 57(2):1157–1188, 2019.

[6] R. Élie, T. Mastrolia and D. Possamaï. A tale of a principal and many, many agents. Mathematics of Operations Research, 44(2):440–467, 2018.

[7] B. Holmström and P. Milgrom. Aggregation and linearity in the provision of intertemporal incentives. Econometrica, 55(2):303–328, 1987.

[8] J. Keppo, N. Touzi and Z. Ruiting. Dynamic contracting in asset management under investor–partner–manager relationship. Preprint SSRN 3761499, 2021.

[9] F. H. Knight. Risk, uncertainty and profit. Boston: Houghton Mifflin Company, The Riverside Press, Cambridge, 1921.

[10] J. A. Mirrlees. The optimal structure of incentives and authority within an organization. The Bell Journal of Economics, 7(1):105–131, 1976.

[11] Y. Sannikov. A continuous–time version of the principal–agent problem. The Review of Economic Studies, 75(3):957–984, 2008.

[12] J. E. Stiglitz. Incentives, risk, and information: notes towards a theory of hierarchy. The Bell Journal of Economics, 6(2):552–579, 1975.

[13] J. Sung. Pay for performance under hierarchical contracting. Mathematics and Financial Economics, 9 (3):195–213, 2015.