Primal return ambiguity & Dual risk ambiguity

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Motivations (1/3)

- What is the **ambiguity** (or uncertainty)?
  $\Rightarrow$ "*...because he has a relatively vague idea as to what the true probability assignment...*" Klibanoff, Marinacci, Mukerji (2005, Econometrica)

- What kind of ambiguity is considered in Finance or Economics?
  $\Rightarrow$ Knightian uncertainty, Epstein and Wang (1994, Econometrica),
  $\Rightarrow$ Smooth ambiguity, Klibanoff, Marinacci, Mukerji (2005, Econometrica),
  $\Rightarrow$ Model parameters' ambiguity, e.g., return, Garlappi, Uppal, Wang, (2007, RFS), volatility, Matoussi, Possamai, Zhou, (2015, MF), correlation, Fouque, Pun, Wong (2016, SICON)

- Why should we consider the ambiguity in **decision making**?
  $\Rightarrow$ To assess risk and derive investment strategy in *robust manner*!!

Motivations (2/3)

The Markowitz Optimization Enigma: Is 'Optimized' Optimal?

• DeMiguel et al. (2009, RFS) : The theoretically optimal strategies with plug-in estimates cannot outperforms (out-of-sample) **equally-weighted (1/p) portfolio**[1].

Table: Monthly Sharpe ratio (The bigger, the better)

Monthly Sharpe ratio Strategy	S&P sectors	Industry portfolios	Inter'l portfolios	FF 1-factor	FF 4-factor
1/p (equally-weighted)	**0.1876**	**0.1353**	**0.1277**	**0.1623**	**0.1753**
MV	0.0794	0.0679	-0.0332	0.0128	0.1841
MV-Constraint	0.0892	0.0678	0.0848	0.1977	0.2024
bs-c (bs-constraint)	0.1075	0.0819	0.0848	0.1955	0.2062
min-c (min-constraint)	0.0834	0.1425	0.1501	0.1546	0.3580

[1] 1/p portfolio invests an equal amount of money into p financial assets in the market.

Motivations (3/3)

- Errors Accumulation

For traditional estimation, the following consequence is shown by Pun and Wong (2016, SIFIN), Chiu, Pun, Wong (2017, Risk Analysis), Pun and Wong (2019, EJOR).

Theorem

If **the estimation error exists** and the portfolio size is so large that $m/n \to \infty$, then

$$\mathbb{P}(\text{Optimal Static Portfolio} > \text{Bank Deposit}) \xrightarrow{P} \frac{1}{2},$$

$$\mathbb{P}(\text{Optimal Dynamic Portfolio} > \text{Bank Deposit}) \leq U_{n,m} \xrightarrow{P} \frac{1}{2},$$

$$\mathbb{P}(\text{Optimal Dynamic Portfolio} > \text{Optimal Static Portfolio}) \xrightarrow{P} \frac{1}{2}.$$

Literature review

- Estimation difficulties with long-term observation in the **return parameter**: Merton (1980, JFE), Cont (2001, QF)

- **Return ambiguity** has been incorporated into finance and economic problems: Britten-Jones (1992, JF), Epstein and Wang (1994, Econometrica), Garlappi, Uppal, Wang, (2007, RFS)

- Robust consumption-investment studies on **the CRRA** setup: Lin, Riedel (2021, ET), Biagini and Pinar, (2017, MAFE)

- **Optimal stopping problem in consumption-investment**
  - Robust retirement decision is yet to be considered in the literature.
    $\Rightarrow$ Dybvig and Liu (2009, JET) : Retirement and constrained borrowing
    $\Rightarrow$ Yang, Koo, Shin (2020, AMO) : Incomplete market (BSPDE approach)
    $\Rightarrow$ Chen, Jeon, Wong (2021, MOR) : Partial information (filtering process)
    $\Rightarrow$ Our work considers a robust stopping problem.

Literature review

- Robust duality theory on **terminal utility maximization** :

  ⇒ Denis and Kervarec (2013,SICON) : Semimartingale framework with some techincal conditions (the bounded utility function, trading strategies as a functional of the stock price, and diagonal form of volatility matrix).

  ⇒ Bartl, Kupper, and Neufeld, (2021,FS) : Semimartingale framework

- Robust stopping time under **the adverse nonlinear expectation** ($\sup_\tau \inf_{\mathbb{P}}$)

⇒ Riedel (2009,Econometrica), Bayraktar and Yao (2011,SPA), Nutz and Zhang (2015,AAP)

- Our dual stopping problem is also related to the **stochastic controller and stopper game**:

⇒ Bayraktar and Huang (2013, SICON)

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Return ambiguity in Primal

- One riskless asset $(S_{0,t})_{t=0}^{T^1}$ with $r > 0$ and two-risky assets $(S_{1,t})_{t=0}^{T^1}$ and $(S_{1,t})_{t=0}^{T^1}$ with *ambiguous drift processes* $(\mu_{1,t})_{t=0}^{T^1}$ and $(\mu_{2,t})_{t=0}^{T^1}$, respectively,

$$(\mu_{1,t}, \mu_{2,t})^\top \in \mathfrak{D} \quad \forall t \in [0, T^1] \quad \text{with} \quad \mathfrak{D} \equiv [\underline{\mu}_1, \bar{\mu}_1] \times [\underline{\mu}_2, \bar{\mu}_2] \subseteq \mathbb{R}^2. \tag{1}$$

- Two risky assets $(d = 2)$ follow an Itô process, i.e.,

$$\begin{aligned} dS_{1,t} &= \mu_{1,t} S_{1,t} dt + \sigma_1 S_{1,t} dW_{1,t}, \\ dS_{2,t} &= \mu_{2,t} S_{2,t} dt + \sigma_2 S_{2,t} (\rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}), \end{aligned} \tag{2}$$

where $W_t^0 = (W_{1,t}^0, W_{2,t}^0)^\top$ is the standard Brownian motion on $(\Omega, \mathcal{F}^0, \mathbb{P}^0)$.

Return ambiguity in Primal

- Denote by Θ the set of all $\mathbb{R}^2-$valued primitive density generators $(\theta_t)_{t=0}^{T^1}$ such that

$$\theta_t \equiv (\theta_{1,t}, \theta_{2,t})^\top \in \mathfrak{R} \quad \forall t \in [0,T] \quad \text{with} \quad \mathfrak{R} \equiv [\underline{\theta}_1, \bar{\theta}_1] \times [\underline{\theta}_2, \bar{\theta}_2] \subseteq \mathbb{R}^2,$$

where $\underline{\theta}_i \equiv \frac{\underline{\mu}_i - r}{\sigma_i}$ and $\bar{\theta}_i \equiv \frac{\bar{\mu}_i - r}{\sigma_i}$, and $(\theta_{i,t})_{t=0}^{T} \in L^2_{\mathbb{F}^0}([0,T^1]; [\underline{\theta}_i, \bar{\theta}_i])$.

$\Rightarrow$ For the set Θ, we can define $\mathcal{P}$ by the set of all priors $\mathbb{P}$ on $(\Omega, \mathcal{F}^0)$ such that

$$\mathcal{P} \equiv \left\{ \mathbb{P} \;\middle|\; \exists\, (\theta_t)_{t=0}^{T^1} \in \Theta \;\text{ s.t. }\; \left.\frac{d\mathbb{P}}{d\mathbb{P}^0}\right|_{\mathcal{F}_t^0} = \xi_t^\theta \right\},$$

where $(\xi_t^\theta)_{t=0}^{T^1}$ is an exponential $\mathbb{P}^0$-martingale, given by

$$\xi_t^\theta = \exp\Big(-\frac{1}{2}\int_0^t \frac{\theta_{1,s}^2 - 2\rho\theta_{1,s}\theta_{2,s} + \theta_{2,s}^2}{1-\rho^2} ds + \int_0^t \theta_{1,s} dW_{1,s}^0 + \int_0^t \frac{-\rho\theta_{1,s} + \theta_{2,s}}{\sqrt{1-\rho^2}} dW_{2,s}^0 \Big).$$

$\Rightarrow$ Under $\mathbb{P}$, the ambiguity-averse investor's wealth is

$$\begin{aligned} dX_t^{c,\pi} &= \big(rX_t^{c,\pi} - c_t\big)dt + \sigma_1 \pi_{1,t}\big(\theta_{1,t}dt + dW_{1,t}\big) \\ &\quad + \sigma_2 \pi_{2,t}\big(\theta_{2,t}dt + \rho dW_{1,t} + \sqrt{1-\rho^2} dW_{2,t}\big). \end{aligned}$$

Robust optimization

- the wealth process $(X_t^{c,\pi})_{t=0}^{T^1}$ satisfies

$$X_t^{c,\pi} \geq 0 \quad \text{for all} \;\; t \in [0, T^1], \quad \mathbb{P} - a.s, \tag{3}$$

- c_t and $\pi_t = (\pi_{1,t}, \pi_{2,t})^\top$ are $\mathcal{F}_t$-progressively measurable processes and satisfy $\int_0^{T^1} (c_t + \|\pi_t\|^2) dt < \infty, \quad \mathbb{P} - a.s$, subject to

$$c_t \geq 0,$$

where $(\mathcal{F}_t)_{t=0}^{T^1}$ is the natural filtration.
$\Rightarrow$ Denote by $\mathcal{A}_{0,T^1}(x)$ the set of all admissible strategies with $x > 0$ over the time horizon $[0, T^1]$.

Primal problem

For a given $x > 0$, determine a robust strategy, $(c^*, \pi^*) \in \mathcal{A}_{0,T^1}(x)$, such that

$$V(0,x) \equiv \sup_{(c,\pi) \in \mathcal{A}_{0,T^1}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\int_0^{T^1} e^{-\delta t} u_1(c_t) dt + e^{-\delta T^1} u_2(X_{T^1}^{c,\pi}) \right]. \tag{4}$$

Risk ambiguity in Dual

- Recall a state-price-density under a fixed prior $\mathbb{P} \in \mathcal{P}$ with $(\theta_t)_{t=0}^{T^1} \in \Theta$, given by

$$e^{-rt} \left. \frac{d\mathbb{P}_0}{d\mathbb{P}} \right|_{\mathcal{F}_t},$$

where

$$\left. \frac{d\mathbb{P}_0}{d\mathbb{P}} \right|_{\mathcal{F}_t} \equiv \exp\Big(-\frac{1}{2} \int_0^t \frac{\theta_{1,s}^2 - 2\rho\theta_{1,s}\theta_{2,s} + \theta_{2,s}^2}{1-\rho^2} ds - \int_0^t \theta_{1,s} dW_{1,s} - \int_0^t \frac{-\rho\theta_{1,s} + \theta_{2,s}}{\sqrt{1-\rho^2}} dW_{2,s} \Big),$$

$\Rightarrow$ Applying Fatou's lemma and Bayes' rule to $e^{-rt} X_t^{c,\pi}$, we obtain the following static budget constraint:

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^{T^1} e^{-rt} \left. \frac{d\mathbb{P}_0}{d\mathbb{P}} \right|_{\mathcal{F}_t} c_t dt + e^{-rT^1} \left. \frac{d\mathbb{P}_0}{d\mathbb{P}} \right|_{\mathcal{F}_{T^1}} X_{T^1}^{c,\pi} \right] \leq x,$$

which is equivalent to the dynamic wealth constraint.

Risk ambiguity in Dual

- We temporarily assume a **fictitious dual economy**.
- Risk-free asset $\mathcal{Z}_{0,t}$ has a constant rate of return equal to $\delta - r$, and the two risky assets $\mathcal{Z}_{1,t}$ and $\mathcal{Z}_{2,t}$ have *risk ambiguity*, $(\theta_{1,t})_{t=0}^{T^1}$ and $(\theta_{2,t})_{t=0}^{T^1}$, in that $(\theta_{1,t}, \theta_{2,t})^\top \in \mathfrak{R} = [\underline{\theta}_1, \bar{\theta}_1] \times [\underline{\theta}_2, \bar{\theta}_2]$.
- The dynamics of the two assets are given by

$$\frac{d\mathcal{Z}_{1,t}}{\mathcal{Z}_{1,t}} \equiv -\theta_{1,t}\left(dW_{1,t} + \frac{-\rho}{\sqrt{1-\rho^2}}dW_{2,t}\right) \quad \frac{d\mathcal{Z}_{2,t}}{\mathcal{Z}_{2,t}} \equiv -\theta_{2,t}\frac{1}{\sqrt{1-\rho^2}}dW_{2,t}, \tag{5}$$

where $(W_t)_{t=0}^{T^1}$ is the $\mathbb{P}$-Brownian motion.

$\Rightarrow$ The dual process $(Y_t)_{t=0}^{T^1}$ is given by

$$Y_t = ye^{(\delta-r)t}\left.\frac{d\mathbb{P}_0}{d\mathbb{P}}\right|_{\mathcal{F}_t}, \tag{6}$$

which can be regarded as the fictitious wealth process,

$$\frac{dY_t}{Y_t} = \frac{d\mathcal{Z}_{0,t}}{\mathcal{Z}_{0,t}} + \frac{d\mathcal{Z}_{1,t}}{\mathcal{Z}_{1,t}} + \frac{d\mathcal{Z}_{2,t}}{\mathcal{Z}_{2,t}}, \tag{7}$$

which induces **risk ambiguity**.

Key idea: Primal return ambiguity & Dual risk ambiguity

- Examine the robust strategy in the presence of return ambiguity on multiple risky assets.

- The **return ambiguity in primal** can be transformed into the **risk (or volatility) ambiguity in dual**.

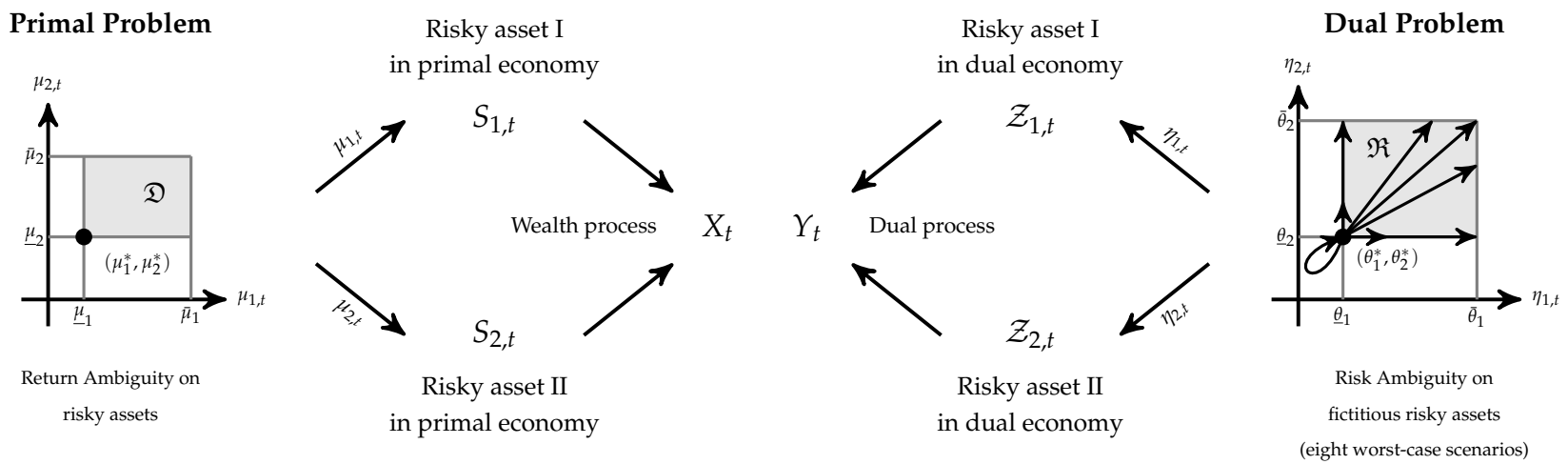


Figure: Primal return ambiguity & Dual risk ambiguity (2-dim risky assets case).

Dual formulation

Using the dual conjugate functions, we have

$$
\begin{aligned}
V(0,x) &= \sup_{(c,\pi)\in\mathcal{A}_{0,T^1}(x)} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-\delta t} u_1(c_t)dt + e^{-\delta T^1} u_2(X_{T^1}^{c,\pi})\right] \\
&\leq \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-\delta t} \sup_{c\geq 0}\left\{u_1(c) - cY_t\right\}dt + e^{-\delta T^1} \sup_{X\geq 0}\left\{u_2(X) - XY_{T^1}\right\}\right] + yx \qquad (8) \\
&= \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-\delta t} \widetilde{u}_1(Y_t)dt + e^{-\delta T^1} \widetilde{u}_2(Y_{T^1})\right] + yx,
\end{aligned}
$$

where the first-order conditions imply that the candidate consumption rate $(c_t^*)_{t=0}^{T^1}$ and bequest $X_{T^1}^*$ are, respectively, given by

$$
c_t^* \equiv I_1(Y_t) \quad \text{and} \quad X_{T^1}^* \equiv I_2(Y_{T^1}). \qquad (9)
$$

where $I_i(y)$ is given by

$$
I_i(y) = \begin{cases} (u_i')^{-1}(y) & \text{if } y \in (0, u_i'(0^+)), \\ 0 & \text{if } y \in [u_i'(0^+), \infty), \end{cases} \qquad (10)
$$

Dual formulation

Dual problem

Consider the following dual optimization problem: for $y > 0$,

$$\underline{J}(0, y) \equiv \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[\int_0^T e^{-\delta t} \widetilde{u}_1(Y_t) dt + e^{-\delta T} \widetilde{u}_2(Y_T) \right], \tag{11}$$

where for each prior $\mathbb{P} \in \mathcal{P}$ with $(\theta_t)_{t=0}^T \in \Theta$, the dual process $(Y_t)_{t=0}^T$ are given by

$$\frac{dY_t}{Y_t} = (\delta - r)dt - \theta_{1,t} \left(dW_{1,t} + \frac{-\rho}{\sqrt{1-\rho^2}} dW_{2,t} \right) - \theta_{2,t} \frac{1}{\sqrt{1-\rho^2}} dW_{2,t}. \tag{12}$$

Model ambiguity: G-Expectation

- We propose a dual approach combining classical dual approach (Karatzas and Wang (2000), SICON) and G-expectation.
  - Hu, Ji, Peng, Song (2014a, SPA) : Existence and Uniqueness of G-BSDE
  - Hu, Ji, Peng, Song (2014b, SPA) : Priori Estimate of G-BSDE and corresponding Feynman Kac formula (under lipschitz condition)
  - Hu, Ji, Yang (2014, AMO) : Stochastic optimization under G-expectation framework
  - Li and Peng (2020, SPA) : Reflected G-BSDE with upper obstacle

Construction for G-expectation

$\Rightarrow$ According to (Denis, Hu, and Peng ('11), Potential Analysis), define the following nonlinear expectation $\hat{\mathbb{E}}$ using the representing prior set $\mathcal{Q}$. For any $\xi \in \mathcal{B}(\Omega)$ such that $\sup_{\widetilde{\mathbb{P}}^{\eta} \in \mathcal{Q}} \mathbb{E}^{\widetilde{\mathbb{P}}^{\eta}}[\xi] < \infty$,

$$\hat{\mathbb{E}}[\xi] \equiv \sup_{\widetilde{\mathbb{P}}^{\eta} \in \mathcal{Q}} \mathbb{E}^{\widetilde{\mathbb{P}}^{\eta}}[\xi]. \tag{13}$$

- The corresponding sublinear function $G : \mathbb{S}(2) \to \mathbb{R}$ is given by

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \operatorname{tr}\left[A\gamma\gamma^{\top}\right], \tag{14}$$

where Γ is the bounded, convex and closed subset of $\mathbb{R}^{2\times 2}$ given by

$$\Gamma \equiv \left\{ \begin{pmatrix} x_1 & -\frac{\rho}{\sqrt{1-\rho^2}} x_1 \\ 0 & \frac{1}{\sqrt{1-\rho^2}} x_2 \end{pmatrix} \in \mathbb{R}^{2\times 2} \middle| (x_1, x_2)^{\top} \in \mathfrak{R} \right\}. \tag{15}$$

- We have defined the G-expectation space $(\Omega, L_G^p(\Omega), \hat{\mathbb{E}})$ in which the dynamics of the dual process in (7) are given by

$$\frac{dY_t}{Y_t} = (\delta - r)dt + dB_{1,t} + dB_{2,t}, \quad t \in [0, T], \tag{16}$$

with G-Brownian motion B and $Y_0 = y > 0$.

Construction for G-expectation

- According to Proposition 3.1.5 in (Peng ('10)), we consider the following 1-dimensional $\widetilde{G}$-Brownian motion. Under $(\Omega, L_G^p(\Omega), \hat{\mathbb{E}})$,

$$\widetilde{B}_t \equiv B_{1,t} + B_{2,t} \quad \text{for all} \quad t \in [0, T], \tag{17}$$

where the corresponding sublinear function $\widetilde{G} : \mathbb{R} \to \mathbb{R}$ is given by

$$\widetilde{G}(\alpha) \equiv \frac{1}{2}\hat{\mathbb{E}}\left[\alpha \widetilde{B}_1^2\right] = \frac{1}{2}\left(\bar{\Upsilon}^2 \alpha^+ - \underline{\Upsilon}^2 \alpha^-\right) \quad \text{for} \quad \alpha \in \mathbb{R}, \tag{18}$$

with $\alpha^{\pm} \equiv \max\{0, \pm\alpha\}$ for all $\alpha \in \mathbb{R}$, $\bar{\Upsilon}^2$ and $\underline{\Upsilon}^2$ are given by

$$\begin{aligned} \bar{\Upsilon}^2 &\equiv 2G\Big((1,1)^\top (1,1)\Big) = \sup_{(x_1,x_2)\in\mathfrak{R}} f(x_1, x_2), \\ \underline{\Upsilon}^2 &\equiv -2G\Big(-(1,1)^\top (1,1)\Big) = \inf_{(x_1,x_2)\in\mathfrak{R}} f(x_1, x_2), \end{aligned} \tag{19}$$

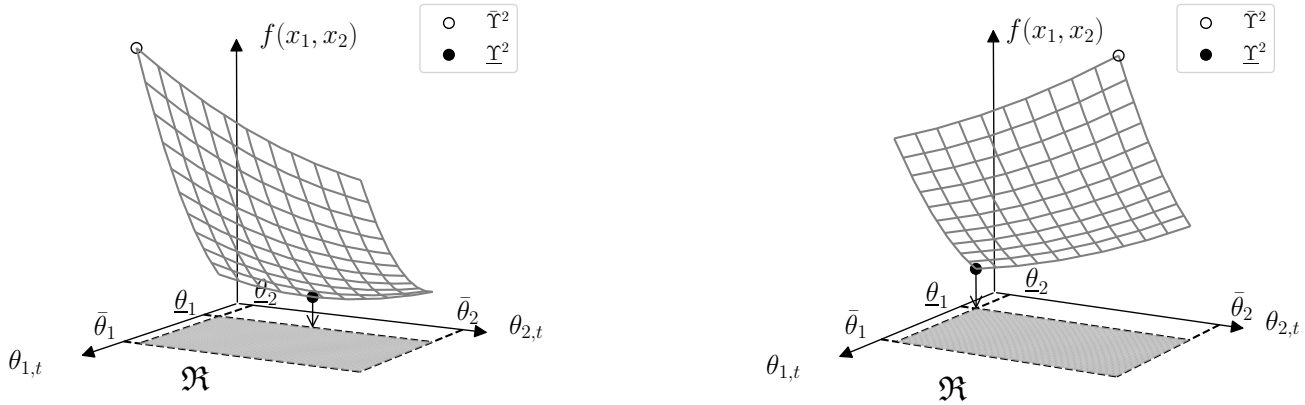
and the two-variable function $f(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x_1, x_2) = \frac{1}{1-\rho^2}\left(x_1^2 - 2\rho x_1 x_2 + x_2^2\right) \quad \text{for} \quad (x_1, x_2) \in \mathbb{R}^2. \tag{20}$$

Construction for G-expectation

$\Rightarrow$ We denote the unique minimizer by $(\theta^*)^\top \equiv (\theta_1^*, \theta_2^*) \in \mathfrak{R}$ such that

$$\underline{\Upsilon}^2 = \inf_{(x_1,x_2)\in\mathfrak{R}} f(x_1, x_2) = f(\theta_1^*, \theta_2^*). \tag{21}$$



(a) Ambiguity in $\theta_t^\top \in \mathfrak{R}$ under $0 < \rho < 1$ (b) Ambiguity in $\theta_t^\top \in \mathfrak{R}$ under $-1 < \rho < 0$

Figure: The variance ambiguity of $\widetilde{B}$ under two correlation cases

Risk ambiguity in dual risky assets

Dual problem under the G-expectation space

Consider the following nonlinear expectation: for $y > 0$,

$$\widetilde{\underline{J}}(t,y) \equiv -\hat{\mathbb{E}}_t\left[-\left(\int_t^{T^1} e^{-\delta(s-t)}\widetilde{u}_1(Y_s)ds + e^{-\delta(T^1-t)}\widetilde{u}_2(Y_{T^1})\right)\Big| Y_t = y\right], \tag{22}$$

where the dual process $(Y_t)_{t=0}^{T^1}$, which is the solution of (16), is given by

$$Y_t = y\exp\left((\delta - r)t - \int_0^t d\widetilde{B}_s - \frac{1}{2}\int_0^t d\langle\widetilde{B}\rangle_s\right), \tag{23}$$

which satisfies $dY_t = (\delta - r)Y_t dt - Y_t d\widetilde{B}_t$ for $t \in [0, T^1]$.

- Consider the (decoupled) forward-backword stochastic differential equation driven by $G-$Brownian motion (G-FBSDE).

$$\begin{aligned} Y_u^{t,y} &= y + \int_t^u (\delta - r)Y_s^{t,y}ds - \int_t^u Y_s^{t,y}d\widetilde{B}_s \quad \text{with} \quad Y_t^{t,y} = y > 0, \\ \underline{\mathcal{G}}_u^{t,y} &= -\widetilde{u}_2(Y_{T^1}^{t,y}) - \int_u^{T^1}\left(\widetilde{u}_1(Y_s^{t,y}) + \delta\underline{\mathcal{G}}_s^{t,y}\right)ds - \int_u^{T^1}\underline{\mathcal{M}}_s^{t,y}d\widetilde{B}_s - (\underline{\mathcal{K}}_{T^1}^{t,y} - \underline{\mathcal{K}}_u^{t,y}). \end{aligned} \tag{24}$$

where $\mathcal{M}$ is a symmetric $G-$martingale, $\mathcal{K}$ is a decreasing G-martingale with $\mathcal{K}_0 = 0$.

G-BSDE

Lemma

(i) G-BSDE (24) has a unique solution in $\mathfrak{S}_G^{\alpha}(0,T^1)$ with some $\alpha > 1$.

(ii) Denote by

$$\underline{g}(t,y) \equiv \underline{\mathcal{G}}_t^{t,y} \quad \text{for} \quad (t,y) \in \overline{\mathcal{D}}_{T^1}.$$

Then, $\underline{g}(t,y)$ is the unique viscosity solution of the following partial differential equation:

$$\begin{cases} \partial_t \underline{g} + \widetilde{G}\left(y^2 \partial_{yy} \underline{g}\right) + (\delta - r) y \partial_y \underline{g} - \delta \underline{g} - \widetilde{u}_1(y) = 0, & (t,y) \in \mathcal{D}_{T^1}, \\ \underline{g}(T^1, y) = -\widetilde{u}_2(y), \quad y \in (0,\infty). \end{cases} \tag{25}$$

$\Rightarrow$ Extend Feynman-Kac formula in Hu, Ji, Peng, Song (2014b, SPA) under the relaxed growth conditions for the driver and terminal function with respect to the forward process Y.

Characterization of the dual problem by G-expectation

Proposition

The dual value function $\underline{J}(0,y)$ in (11) is the same as the auxiliary problem $\widetilde{\underline{J}}(0,y)$ and the prior $\mathbb{P}^*$ realizes the worst-case scenario on the dual problem, i.e.,

$$\begin{aligned}\underline{J}(0,y) &= \mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1} e^{-\delta t}\widetilde{u}_1(Y_t)dt + e^{-\delta T^1}\widetilde{u}_2(Y_{T^1})\right] \\ &= -\hat{\mathbb{E}}\left[-\left(\int_0^{T^1} e^{-\delta t}\widetilde{u}_1(Y_t)dt + e^{-\delta T^1}\widetilde{u}_2(Y_{T^1})\right)\right] = \widetilde{\underline{J}}(0,y),\end{aligned} \tag{26}$$

where $\mathbb{P}^* \in \mathcal{P}$ with corresponding generator $\theta^* = (\theta_1^*, \theta_2^*)^\top$ in (21) and $(Y_t)_{t=0}^{T^1}$ is represented by $Y_t = ye^{(\delta-r)t}\left.\frac{d\mathbb{P}_0}{d\mathbb{P}^*}\right|_{\mathcal{F}_t}, \quad t \in [0, T^1]$, where

$$\left.\frac{d\mathbb{P}_0}{d\mathbb{P}^*}\right|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\underline{Y}^2 t - \theta_1^* W_{1,t}^* - \frac{-\rho\theta_1^* + \theta_2^*}{\sqrt{1-\rho^2}} W_{2,t}^*\right), \tag{27}$$

with $\mathbb{P}^*$-Brownian motion $(W_t^*)_{t=0}^{T^1}$.

Verification and Duality Theorem

Duality theorem and optimal strategies

For a given $x > 0$, the value function V in (35) and dual value function J in (26) satisfy the following duality relationship:

$$V(0,x) = \min_{y>0} \left(\underline{J}(0,y) + yx \right) = \underline{J}(0,y^*) + y^* x, \tag{28}$$

where $y^* \equiv I_J(-x) > 0$ is the unique minimizer, and $I_J(\cdot)$ is the inverse function of $\partial_y J(0,\cdot)$.
Furthermore, the value function V is represented by

$$V(0,x) = \mathbb{E}^{\mathbb{P}^*} \left[\int_0^{T^1} e^{-\delta t} u_1(c_t^*) dt + e^{-\delta T^1} u_2(X_{T^1}^{c^*,\pi^*}) \right], \tag{29}$$

where c^*, π^*, and X^{c^*,π^*} are the robust consumption, risky investment, and wealth of the ambiguity averse investor, given by

$$c_t^* = I_1(Y_t^*), \quad \pi_t^* = \frac{1}{1-\rho^2} Y_t^* \partial_{yy} \underline{J}(t, Y_t^*) \left(\frac{\theta_1^*}{\sigma_1} - \rho \frac{\theta_2^*}{\sigma_1}, \ -\rho \frac{\theta_1^*}{\sigma_2} + \frac{\theta_2^*}{\sigma_2} \right)^\top, \tag{30}$$

and

$$X_t^{c^*,\pi^*} = -\partial_y \underline{J}(t, Y_t^*), \quad t \in [0, T^1], \tag{31}$$

respectively, where $Y_t^* \equiv y^* e^{(\delta - r)t} \left. \frac{d\mathbb{P}_0}{d\mathbb{P}^*} \right|_{\mathcal{F}_t}$ for $t \in [0, T^1]$ with the minimizer $y^* > 0$.

Applications (2/2)

Corollary 2: the CARA

For a given $x > 0$, the primal value function $V(0,x)$ can be represented by

$$V(0,x) = J(0,y^*) + y^* x,$$

and the dual value function $J(0,y)$ is given by

$$\begin{aligned} J(0,y) = &\int_0^T e^{-\delta t}\mathcal{N}\Big(-d^-\Big(t,\frac{y}{\beta}\Big)\Big)dt + e^{-\delta T}\mathcal{N}\Big(-d^-\Big(T,\frac{y}{\beta k_b}\Big)\Big) \\ &- \int_0^T e^{-rt}\frac{y}{\beta}\left[\Big(1-\log(\frac{y}{\beta}) - (\delta - r - \frac{1}{2}\underline{\Upsilon}^2)t\Big)\mathcal{N}\Big(-d^+\Big(t,\frac{y}{\beta}\Big)\Big) + \underline{\Upsilon}\mathbf{n}\Big(d^+\Big(t,\frac{y}{\beta}\Big)\Big)\right]dt \\ &- e^{-rT}\frac{y}{\beta k_b}\left[\Big(1-\log(\frac{y}{\beta k_b}) - (\delta - r - \frac{1}{2}\underline{\Upsilon}^2)T\Big)\mathcal{N}\Big(-d^+\Big(T,\frac{y}{\beta k_b}\Big)\Big) + \underline{\Upsilon}\mathbf{n}\Big(d^+\Big(T,\frac{y}{\beta k_b}\Big)\Big)\right], \end{aligned}$$

where $\mathcal{N}(\cdot)$ and $\mathbf{n}(\cdot)$ are the standard normal cumulative distribution function and probability density function, respectively, $d^{\pm}(t,z)$ is given by

$$d^{\pm}(t,z) \equiv \frac{\log(z) + (\delta - r \pm \frac{1}{2}\underline{\Upsilon}^2)t}{\underline{\Upsilon}\sqrt{t}} \quad \text{for} \quad (t,z) \in [0,T] \times (0,\infty), \tag{32}$$

and the unique minimizer $y^* > 0$ satisfies

$$\begin{aligned} x = -\partial_y J(0,y^*) = &-\int_0^T e^{-rt}\frac{1}{\beta}\left[\Big(\log(\frac{y^*}{\beta}) + (\delta - r + \frac{1}{2}\underline{\Upsilon}^2)t\Big)\mathcal{N}\Big(-d^+\Big(t,\frac{y^*}{\beta}\Big)\Big) - \underline{\Upsilon}\mathbf{n}\Big(d^+\Big(t,\frac{y^*}{\beta}\Big)\Big)\right]dt \\ &- e^{-rT}\frac{1}{\beta k_b}\left[\Big(\log(\frac{y^*}{\beta k_b}) + (\delta - r + \frac{1}{2}\underline{\Upsilon}^2)T\Big)\mathcal{N}\Big(-d^+\Big(T,\frac{y^*}{\beta k_b}\Big)\Big) - \underline{\Upsilon}\mathbf{n}\Big(d^+\Big(T,\frac{y^*}{\beta k_b}\Big)\Big)\right]. \end{aligned}$$

Applications (2/2)

Corollary 2: the CARA utility function (cont')

The robust consumption and risky investment are given as follows: for $t \in [0, T]$

$$c_t^* = -\frac{1}{\beta}\ln\left(\frac{Y_t^*}{\beta}\right)\mathbf{1}_{\{0<Y_t^*<\beta\}}, \quad \pi_t^* = \frac{1}{1-\rho^2}Y_t^*\partial_{yy}J(t, Y_t^*)\left(\frac{\theta_1^*}{\sigma_1} - \rho\frac{\theta_2^*}{\sigma_1}, -\rho\frac{\theta_1^*}{\sigma_2} + \frac{\theta_2^*}{\sigma_2}\right)^\top,$$

where $Y_t^*\partial_{yy}J(t, Y_t^*)$ is given by

$$\begin{aligned}
Y_t^*\partial_{yy}J(t, Y_t^*) = &-\int_t^T e^{-r(u-t)}\frac{1}{\beta}\Bigg[\mathcal{N}\Big(-d^+\Big(u-t, \frac{Y_t^*}{\beta}\Big)\Big) \\
&+ \Big(-\log(\frac{Y_t^*}{\beta}) - (\delta - r + \frac{1}{2}\underline{Y}^2)(u-t) + \underline{Y}d^+\Big(u-t, \frac{Y_t^*}{\beta}\Big)\Big)\frac{1}{\underline{Y}\sqrt{u-t}}\mathbf{n}\Big(d^+\Big(u-t, \frac{Y_t^*}{\beta}\Big)\Big)\Bigg]du \\
&- e^{-r(T-t)}\frac{1}{\beta k_b}\Bigg[\mathcal{N}\Big(-d^+\Big(T-t, \frac{Y_t^*}{\beta k_b}\Big)\Big) \\
&+ \Big(-\log(\frac{Y_t^*}{\beta k_b}) - (\delta - r + \frac{1}{2}\underline{Y}^2)(T-t) + \underline{Y}d^+\Big(T-t, \frac{Y_t^*}{\beta k_b}\Big)\Big)\frac{1}{\underline{Y}\sqrt{T-t}}\mathbf{n}\Big(d^+\Big(T-t, \frac{Y_t^*}{\beta k_b}\Big)\Big)\Bigg],
\end{aligned}$$

and the robust wealth is given as follows:

$$\begin{aligned}
X_t^* &= -\partial_y J(t, Y_t^*) \\
&= -\int_t^T e^{-r(u-t)}\frac{1}{\beta}\Bigg[\Big(\log(\frac{Y_t^*}{\beta}) + (\delta - r + \frac{1}{2}\underline{Y}^2)(u-t)\Big)\mathcal{N}\Big(-d^+\Big(u-t, \frac{Y_t^*}{\beta}\Big)\Big) - \underline{Y}\mathbf{n}\Big(d^+\Big(u-t, \frac{Y_t^*}{\beta}\Big)\Big)\Bigg]du \\
&\quad - e^{-r(T-t)}\frac{1}{\beta k_b}\Bigg[\Big(\log(\frac{Y_t^*}{\beta k_b}) + (\delta - r + \frac{1}{2}\underline{Y}^2)(T-t)\Big)\mathcal{N}\Big(-d^+\Big(T-t, \frac{Y_t^*}{\beta k_b}\Big)\Big) - \underline{Y}\mathbf{n}\Big(d^+\Big(T-t, \frac{Y_t^*}{\beta k_b}\Big)\Big)\Bigg].
\end{aligned}$$

Robust investment in Dual: Sparse Portfolio

• This new insight links our dual formulation with risk ambiguity to **the correlation ambiguity** problem for utility maximization.
$\Rightarrow$ Fouque, Pun, Wong (2016, SICON)

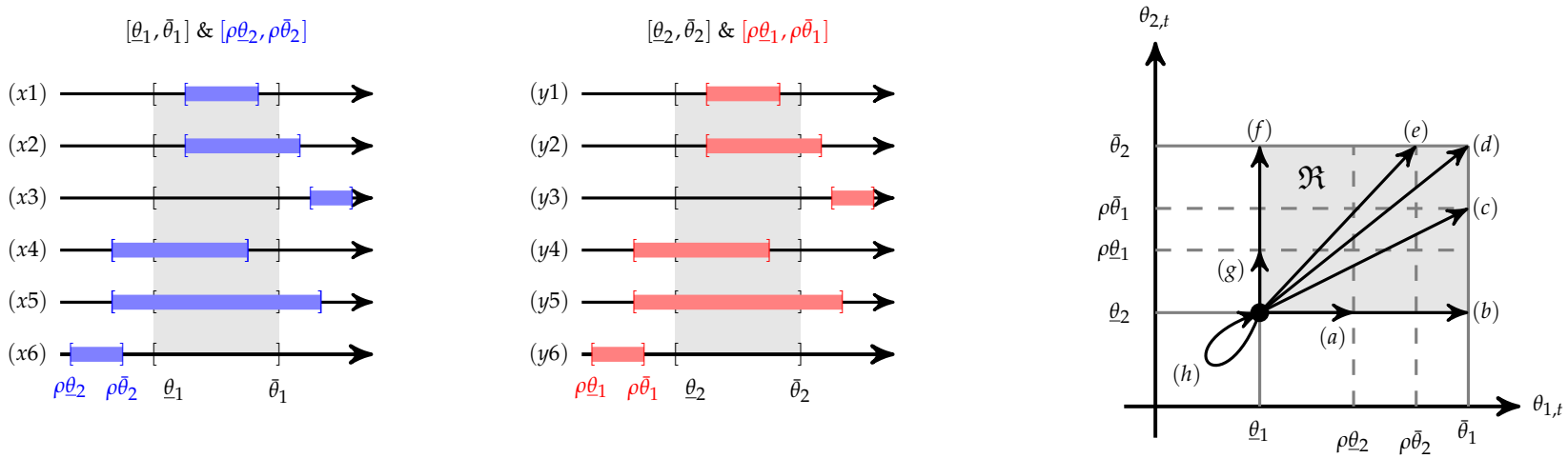
• When $-\underline{\rho} = \overline{\rho} = 1$, the robust optimal portfolio suggests to invest into either one of the risky assets, resulting in a **sparse portfolio**. Related works are:

- Human behavior and correlation ambiguity: Epstein and Halevy (2019, RES)
- Precommitment MV portfolio: Ismail and Pham (2018, MF)

$\Rightarrow$ Robust investment with return ambiguity implies **sparse portfolio**.

Robust strategy: Sparse portfolio (1/2)

$\Rightarrow$ We first consider the case in which the region $\mathfrak{R}$ belongs to the positive real plane and the two risky assets are positively correlated.



Panel a) $\theta_{1,t}$ axis

Panel b) $\theta_{2,t}$ axis

Panel c) 8-worst scenarios

Figure: All possible worst-scenarios under Case 1.1., i.e. $\mathfrak{R} \subset \mathbb{R}_+ \times \mathbb{R}_+$ and $0 \leq \rho < 1$.

Robust strategy: Sparse portfolio (2/2)

$\Rightarrow$ The following equation provides both the optimal investment strategy $(\pi^*_{s,1}, \pi^*_{s,2})$ and the minimum value $\underline{Y}^2$ for all 8 worst scenarios, which implies sparse risky portfolio, i.e.,

$$(\pi^*_{s,1}, \pi^*_{s,2}), \quad \underline{Y}^2 = \begin{cases} Y^*_s \partial_{yy} J(s, Y^*_s)\left(0, \frac{\underline{\theta}_2}{\sigma_2}\right), \quad \underline{\theta}_2^2 & (a): (\rho\underline{\theta}_2, \underline{\theta}_2), \\ \frac{Y^*_s \partial_{yy} J(s, Y^*_s)}{1-\rho^2}\left(\frac{\bar{\theta}_1}{\sigma_1} - \rho\frac{\underline{\theta}_2}{\sigma_1}, -\rho\frac{\bar{\theta}_1}{\sigma_2} + \frac{\underline{\theta}_2}{\sigma_2}\right), \quad \frac{1}{1-\rho^2}(\bar{\theta}_1^2 - 2\rho\bar{\theta}_1\underline{\theta}_2 + \underline{\theta}_2^2) & (b): (\bar{\theta}_1, \underline{\theta}_2), \\ Y^*_s \partial_{yy} J(s, Y^*_s)\left(\frac{\bar{\theta}_1}{\sigma_1}, 0\right), \quad \bar{\theta}_1^2 & (c): (\bar{\theta}_1, \rho\bar{\theta}_1), \\ \frac{Y^*_s \partial_{yy} J(s, Y^*_s)}{1-\rho^2}\left(\frac{\bar{\theta}_1}{\sigma_1} - \rho\frac{\bar{\theta}_2}{\sigma_1}, -\rho\frac{\bar{\theta}_1}{\sigma_2} + \frac{\bar{\theta}_2}{\sigma_2}\right), \quad \frac{1}{1-\rho^2}(\bar{\theta}_1^2 - 2\rho\bar{\theta}_1\bar{\theta}_2 + \bar{\theta}_2^2) & (d): (\bar{\theta}_1, \bar{\theta}_2), \\ Y^*_s \partial_{yy} J(s, Y^*_s)\left(0, \frac{\bar{\theta}_2}{\sigma_2}\right), \quad \bar{\theta}_2^2 & (e): (\rho\bar{\theta}_2, \bar{\theta}_2), \\ \frac{Y^*_s \partial_{yy} J(s, Y^*_s)}{1-\rho^2}\left(\frac{\underline{\theta}_1}{\sigma_1} - \rho\frac{\bar{\theta}_2}{\sigma_1}, -\rho\frac{\underline{\theta}_1}{\sigma_2} + \frac{\bar{\theta}_2}{\sigma_2}\right), \quad \frac{1}{1-\rho^2}(\underline{\theta}_1^2 - 2\rho\underline{\theta}_1\bar{\theta}_2 + \bar{\theta}_2^2) & (f): (\underline{\theta}_1, \bar{\theta}_2), \\ Y^*_s \partial_{yy} J(s, Y^*_s)\left(\frac{\underline{\theta}_1}{\sigma_1}, 0\right), \quad \underline{\theta}_1^2 & (g): (\underline{\theta}_1, \rho\underline{\theta}_1), \\ \frac{Y^*_s \partial_{yy} J(s, Y^*_s)}{1-\rho^2}\left(\frac{\underline{\theta}_1}{\sigma_1} - \rho\frac{\underline{\theta}_2}{\sigma_1}, -\rho\frac{\underline{\theta}_1}{\sigma_2} + \frac{\underline{\theta}_2}{\sigma_2}\right), \quad \frac{1}{1-\rho^2}(\underline{\theta}_1^2 - 2\rho\underline{\theta}_1\underline{\theta}_2 + \underline{\theta}_2^2) & (h): (\underline{\theta}_1, \underline{\theta}_2). \end{cases}$$

Sparse portfolio with 8-worst-case scenarios

⇒ First, the portfolio strategy that minimizes exposure to uncertainty is robust to the agent, regardless of the scenarios (Dow and da Costa Werlang ('92, Econometrica)).

⇒ The classical Markowitz mean-variance framework under the drift uncertainty, including Brodie et al. ('09, PNAS), Britten-Jones ('99, JF), and Garlappi et al. ('07, RFS) .

Case 1.1	$(x1)$	$(x2)$	$(x3)$	$(x4)$	$(x5)$	$(x6)$
$(y1)$	(a), (g)	(a), (g)	·	(g)	(d), (g)	(g)
$(y2)$	(a), (g)	(a), (g)	·	(g)	·	(g)
$(y3)$	·	·	·	·	·	(f), (h)
$(y4)$	(a)	(a)	·	(c), (e), (h)	(c), (h)	(c), (h), (f)
$(y5)$	(a), (d)	·	·	(e), (h)	·	(d), (f), (h)
$(y6)$	(a)	(a)	(b), (h)	(b), (e), (h)	(b), (d), (h)	(b), (f), (h)

Table: Characterization of the worst case scenarios under Case 1.1 (Note that blanks indicate the case in which a combination of relations from each panel is not possible).

Table of Content

Robust optimization with control and stopping

- For the set Θ, we define $\mathcal{P}$ by the set of all priors $\mathbb{P}$ on $(\Omega_{T^1}, \mathcal{F}^0)$ such that

$$\mathcal{P} \equiv \left\{ \mathbb{P} \;\middle|\; \exists\, (\nu_t)_{t=0}^{T^1} \in \Theta \;\text{ s.t. }\; \left.\frac{d\mathbb{P}}{d\mathbb{P}^0}\right|_{\mathcal{F}_t} = e^{\int_0^t (\Sigma^{-1}\nu_s)^\top dW_s^0 - \frac{1}{2}\int_0^t \|\Sigma^{-1}\nu_s\|^2 ds} \right\}. \quad (33)$$

- For a given $\mathbb{P} \in \mathcal{P}$ with $(\nu_t)_{t=0}^{T^1} \in \Theta$, the price dynamics of the risky assets are uniquely represented as follows: for $t \in [0, T^1]$

$$dS_{i,t} = (\nu_{i,t} + r)S_{i,t}dt + \sum_{j=1}^{d} \sigma^{ij} S_{i,t} dW_{j,t}, \quad (i = 1, 2, \cdots, d),$$

where $(W_t)_{t=0}^{T^1}$ denotes the $\mathbb{P}$-Brownian motion, given by $W_t \equiv W_t^0 - \int_0^t \Sigma^{-1}\nu_s ds$.

- The wealth dynamics $(X_t^{\tau,c,\pi})_{t=0}^{T^1}$ with an initial endowment $X_0^{\tau,c,\pi} = x$: for $t \in [0, T^1]$

$$dX_t^{\tau,c,\pi} = \left(rX_t^{\tau,c,\pi} - c_t + w\mathbf{1}_{\{t \le \tau\}}\right)dt + \pi_t^\top \Sigma\left(\Sigma^{-1}\nu_t dt + dW_t\right). \quad (34)$$

Robust optimization with control and stopping

Problem: Max-Min problem

Determine a robust strategy, $(\tau^*, c^*, \pi^*) \in \mathcal{A}_{0,T^1}(x)$,

$$V(0,x) = \sup_{(\tau,c,\pi)\in\mathcal{A}_{0,T^1}(x)} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-\delta t}\left(u_1(c_t) - l\mathbf{1}_{\{t\le\tau\}}\right)dt + e^{-\rho T^1}u_2(X_{T^1}^{\tau,c,\pi})\right].$$

- Considering the static constraint for all priors in $\mathcal{P}$,

$$\begin{aligned} &= \sup_{\tau\in\mathcal{T}_{0,T}} \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-\delta t}\left(\tilde{u}_1(Y_t) + (wY_t - l)\mathbf{1}_{\{t\le\tau\}}\right)dt + e^{-\delta T^1}\tilde{u}_2(Y_{T^1})\right] + yx \\ &\equiv \sup_{\tau\in\mathcal{T}_{0,T}} \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{J}(0,y;\mathbb{P},\tau) + yx. \end{aligned} \tag{35}$$

Dual problem: Max-Min stopping time problem

Consider the following stochastic control game:

$$J(0,y) = \sup_{\tau\in\mathcal{T}_{0,T}} \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{J}(0,y;\mathbb{P},\tau),$$

where for each prior $\mathbb{P}\in\mathcal{P}$ with $(\nu_t)_{t=0}^T \in \Theta$, the dual process satisfy

$$dY_t = (\delta - r)Y_t dt - (\Sigma^{-1}\nu_t)^\top Y_t dW_t. \tag{36}$$

G-optimal stopping problem

Auxiliary problem 2 (G-optimal stopping time in dual)

Consider the following stopping problem:

$$\widetilde{J}(t,y) = -\inf_{\widetilde{\tau}\in\widetilde{\mathcal{T}}_{t,T}} \hat{\mathbb{E}}_t\left[-\int_t^{\widetilde{\tau}} e^{-\delta(s-t)}\left(\widetilde{u}_1(Y_s)+wY_s-l\right)ds - e^{-\delta(\widetilde{\tau}-t)}\underline{\widetilde{J}}(\widetilde{\tau},Y_{\widetilde{\tau}})\right], \tag{37}$$

where $\widetilde{\mathcal{T}}_{t,T}$ denotes the collection of all G-stopping times having values in $[t,T]$ and $\mathcal{D}_T \equiv [0,T)\times(0,+\infty)$ and $\overline{\mathcal{D}}_T \equiv [0,T]\times(0,+\infty)$.

$\Rightarrow$ A typical example of the G-stopping time is **the first exit time** for a right continuous process, Li and Peng (2020,SPA).

$\Rightarrow$ As $\widetilde{J}$ is an optimal stopping problem under adverse nonlinear expectation (i.e., $\inf_{\widetilde{\tau}}\sup_{\widetilde{\mathbb{P}}}$), a reflected G-BSDE with *upper obstacle* is considered.

Reflected G-BSDE (1/6)

- Consider the following reflected G-(forward) BSDE: for $u \in [t, T]$,

$$\begin{aligned} Y_u^{t,y} &= y + \int_t^u (\delta - r) Y_s^{t,y} ds - \int_t^u Y_s^{t,y} d\widetilde{B}_s \quad \text{with} \quad Y_t^{t,y} = y > 0, \\ \mathcal{G}_u^{t,y} &= \underline{\mathcal{G}}_T^{t,y} - \int_u^T \left(\widetilde{u}_1(Y_s^{t,y}) + w Y_s^{t,y} - l + \delta \mathcal{G}_s^{t,y}\right) ds - \int_u^T \mathcal{M}_s^{t,y} d\widetilde{B}_s + (\mathcal{K}_T^{t,y} - \mathcal{K}_u^{t,y}), \end{aligned} \tag{38}$$

and the upper obstacle is represented by

$$\underline{\mathcal{G}}_u^{t,y} = \underline{\mathcal{G}}_t^{t,y} + \int_t^u b_s^{t,y} ds + \int_t^u l_s^{t,y} d\langle \widetilde{B} \rangle_s + \int_t^u \sigma_s^{t,y} d\widetilde{B}_s, \tag{39}$$

with $\underline{\mathcal{G}}_t^{t,y} = -\underline{\widetilde{J}}(t, y)$,

$$\begin{aligned} b_s^{t,y} &\equiv -\partial_t \underline{\widetilde{J}}(s, Y_s^{t,y}) - (\delta - r) Y_s^{t,y} \partial_y \underline{\widetilde{J}}(s, Y_s^{t,y}), \\ l_s^{t,y} &\equiv -\frac{1}{2} (Y_s^{t,y})^2 \partial_{yy} \underline{\widetilde{J}}(s, Y_s^{t,y}), \quad \text{and} \quad \sigma_s^{t,y} \equiv Y_s^{t,y} \partial_y \underline{\widetilde{J}}(s, Y_s^{t,y}). \end{aligned} \tag{40}$$

Reflected G-BSDE (2/6)

Definition: Li and Peng (2020, SPA)

A triple of processes $(\mathcal{G}_u^{t,y}, \mathcal{M}_u^{t,y}, \mathcal{K}_u^{t,y})_{u=t}^T$ is called a solution to (38) if for some $2 \leq \alpha < \beta$ the following properties are satisfied:

(a) $(\mathcal{G}_u^{t,y}, \mathcal{M}_u^{t,y}, \mathcal{K}_u^{t,y})_{u=t}^T \in \mathbb{S}_G^\alpha(0,T^1)$, i.e., $(\mathcal{G}_u^{t,y})_{u=t}^T \in S_G^\alpha(0,T^1)$, $(\mathcal{M}_u^{t,y})_{u=t}^T \in H_G^\alpha(0,T^1)$, and $(\mathcal{K}_u^{t,y})_{u=t}^T \in S_G^\alpha(0,T^1)$ is a continuous process with finite variation satisfying $\mathcal{K}_t^{t,y} = 0$, and $(-\mathcal{K}_u^{t,y})_{u=t}^T$ is a G-submartingale;

(b) $\mathcal{G}_u^{t,y} \leq \underline{\mathcal{G}}_u^{t,y}$ for all $u \in [t,T]$;

(c) $\mathcal{G}_u^{t,y} = \underline{\mathcal{G}}_T^{t,y} - \int_u^T \left(\widetilde{u}_1(Y_s^{t,y}) + wY_s^{t,y} - l + \delta\mathcal{G}_s^{t,y}\right)ds - \int_u^T \mathcal{M}_s^{t,y} d\widetilde{B}_s + (\mathcal{K}_T^{t,y} - \mathcal{K}_u^{t,y})$ for $u \in [t,T]$;

(d) $(-\int_t^u (\underline{\mathcal{G}}_s^{t,y} - \mathcal{G}_s^{t,y}) d\mathcal{K}_s^{t,y})_{u=t}^T$ is a non-increasing $G-$martingale.

The solution is called a **maximal solution** if $(\mathcal{G}'_u, \mathcal{M}'_u, \mathcal{K}'_u)_{u=t}^T$ is another solution, then $\mathcal{G}_u^{t,y} \geq \mathcal{G}'_u$ for all $u \in [t,T]$.

Reflected G-BSDE (3/6)

Proposition

(i) Reflected G-BSDE (38) has a **maximal solution** in $\mathbb{S}_G^{\alpha}(0, T^1)$ with some $\alpha \geq 2$. Define a function $g(t, y)$ by

$$g(t, y) \equiv \mathcal{G}_t^{t,y} \quad \text{for} \quad (t, y) \in \overline{\mathcal{D}}_T.$$

• Verify the strong regularity and some estimates of the solution $\underline{\widetilde{J}} = -\underline{g}$ (i.e., after retirement dual value function).
$\Rightarrow$ the existence of a maximal solution by Li and Peng (2020, SPA)

Reflected G-BSDE (4/6)

Proposition (cont'

(ii) Then, $g(t,y)$ is the unique viscosity solution of the following **obstacle problem**:

$$\begin{cases} \min\left\{\partial_t g + \widetilde{G}\left(y^2\partial_{yy}g\right) + (\delta - r)y\partial_y g - \delta g - \widetilde{u}_1(y) - wy + l, \underline{g} - g\right\} = 0 \ \text{ in } \ \mathcal{D}_T, \\ g(T,y) = \underline{g}(T,y), \quad y \in (0,\infty). \end{cases}$$

• Demonstrate the priori estimates of reflected G-BSDE with the upper obstacle by using *approximate Skorohod condition*, which is proposed to obtain estimates of doubly reflected G-BSDE with lower and upper obstacles Li and Song (2019, working).

$\Rightarrow$ We can prove the continuity of the reflected G-BSDE with upper obstacle.

$\Rightarrow$ the notion of a viscosity solution Crandall, Ishii, Lions (1992, BAMS)

Reflected G-BSDE (5/6)

Proposition

(i) $\widetilde{J} \in W^{2,1}_{p,loc}(\mathcal{D}_T) \cap C(\overline{\mathcal{D}}_T)$ for any $p \geq 1$, $\partial_t \widetilde{J}, \partial_y \widetilde{J} \in C(\overline{\mathcal{D}}_T)$, and $\partial_{yy}\widetilde{J} > 0$ a.e. in $\overline{\mathcal{D}}_T$. Moreover, $\widetilde{J}$ satisfies the following lower obstacle linear PDE:

$$\begin{cases} \max\left\{\partial_t \widetilde{J} + \mathcal{L}_{\underline{\Upsilon}}\widetilde{J} + \widetilde{u}_1(y) + wy - l, \underline{\widetilde{J}} - \widetilde{J}\right\} = 0, & (t,y) \in \mathcal{D}_T, \\ \widetilde{J}(T,y) = \underline{\widetilde{J}}(T,y), \quad y \in (0,\infty), \end{cases} \tag{41}$$

with the operator

$$\mathcal{L}_{\underline{\Upsilon}} \equiv \frac{1}{2}\underline{\Upsilon}^2 y^2 \partial_{yy} + (\delta - r) y \partial_y - \delta.$$

Thus, this implies that $\widetilde{J}$ is a classical optimal stopping value under $\widetilde{\mathbb{P}}^{\nu^*} \in \mathcal{Q}$ with $M^{\nu^*} = \mathrm{diag}(\nu^*)(\Sigma^{-1})^\top$.

$\Rightarrow$ The unique viscosity solution is actually strong solution of the variational inequality in Yang and Koo (2018, MOR)

$\Rightarrow$ The convexity of $\widetilde{J}$ and the uniqueness enable the upper obstacle reflected G-BSDE to be represented as the classical variational inequality.

Reflected G-BSDE (6/6)

Proposition (cont')

(ii) Optimal G-stopping time $\tau^* \in \widetilde{\mathcal{T}}_{0,T}$ is given by

$$\tau^* = \inf\left\{t \in [0,T] \mid Y_t \leq \hat{z}_{\nu^*}(t)\right\}, \tag{42}$$

where a free boundary $(\hat{z}_{\nu^*}(t))_{t=0}^{T}$, which is in $C^\infty([0,T)) \cap C([0,T])$, is deterministic and time-dependent and satisfies the following integral equation:

$$\begin{aligned} 0 = w\hat{z}_{\nu^*}(t) \int_t^T e^{-r(s-t)} \mathcal{N}\Big(d^+\big(s-t, \frac{\hat{z}_{\nu^*}(t)}{\hat{z}_{\nu^*}(s)}\big)\Big) ds \\ - l \int_t^T e^{-\delta(s-t)} \mathcal{N}\Big(d^-\big(s-t, \frac{\hat{z}_{\nu^*}(t)}{\hat{z}_{\nu^*}(s)}\big)\Big) ds, \end{aligned} \tag{43}$$

where $\mathcal{N}(\cdot)$ is the standard normal cumulative distribution and $d^{\pm}(t,x) \equiv \frac{\log(x)+(\delta-r\pm\frac{1}{2}\underline{Y}^2)t}{|\underline{Y}|\sqrt{t}}$.

$\Rightarrow$ G-optimal stopping time is characterized by closed and regular free boundary $\hat{z}_{\nu^*}(s)$ satisfying the deterministic integral equation.

Verification and Duality (1/2)

Theorem

The solution $J(0,y)$ to the original dual problem is the same as $\widetilde{J}(0,y)$, i.e.,

$$J(0,y) = \sup_{\tau\in\mathcal{T}_{0,T}} \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{J}(0,y;\mathbb{P},\tau)$$

$$= -\inf_{\widetilde{\tau}\in\widetilde{\mathcal{T}}_T} \hat{\mathbb{E}}\left[-\int_0^{\widetilde{\tau}} e^{-\delta t}\left(\widetilde{u}_1(Y_t) + wY_t - l\right)dt - e^{-\delta\widetilde{\tau}}\underline{\widetilde{J}}(\widetilde{\tau}, Y_{\widetilde{\tau}})\right] = \widetilde{J}(0,y)$$

in which the prior $\mathbb{P}^*$ with $(\nu_t^*)_{t=0}^{T^1}$ in (21) and the stopping time τ^* in (42) realize the worst-case optimal stopping scenario on original dual problem, i.e.,

$$J(0,y) = \mathcal{J}(0,y;\mathbb{P}^*,\tau^*).$$

$\Rightarrow$ We utilize the following equivalence between two priors $\mathbb{P}^* \in \mathcal{P}$ and $\widetilde{\mathbb{P}}^{\nu^*} \in \mathcal{Q}$, i.e.,

$$\mathrm{Law}(\mathbb{P}^*) = \mathrm{Law}(\widetilde{\mathbb{P}}^{\nu^*}), \qquad \left.\frac{d\mathbb{P}^0}{d\mathbb{P}^*}\right|_{\mathcal{F}_t} = e^{-\int_0^t \left(\Sigma^{-1}\nu^*\right)^\top dW_s - \frac{1}{2}\int_0^t \|\Sigma^{-1}\nu^*\|^2 ds}$$

$\Rightarrow$ This implies that the stochastic controller and stopper game (original dual problem) on **dominated priors** is equivalent to the optimal stopping problem under adverse nonlinear expectation on **non-dominated priors**!

Verification and Duality (2/2)

Theorem

(i) For a given $x > 0$, the value function V and dual value function J satisfy the following duality relationship:

$$V(0,x) = \min_{y>0} \Big(J(0,y) + yx \Big) = J(0,y^*) + y^* x, \tag{44}$$

where $y^* \equiv I_J(-x) > 0$.

(ii) Furthermore, where $\tau^* = \tau^*(y^*)$ is the robust retirement time, given by

$$\tau^*(y^*) = \inf \big\{ t \in [0,T] \mid Y_t^* \leq \hat{z}_{\nu^*}(t) \big\}, \tag{45}$$

$(c_t^*)_{t=0}^{T1}$, $(\pi_t^*)_{t=0}^{T1}$, and $(X_t^{\tau^*,c^*,\pi^*})_{t=0}^{T1}$ are the robust consumption, risky investment, and wealth of the investor, given by

$$\begin{aligned} &c_t^*(y^*) = I_1(Y_t^*), \quad \pi_t^*(y^*) = Y_t^* \partial_{yy} J(t, Y_t^*)(\Sigma^\top)^{-1}\Sigma^{-1}\nu^*, \quad \text{and} \\ &X_t^{\tau^*,c^*,\pi^*}(y^*) = -\partial_y J(t, Y_t^*), \end{aligned} \tag{46}$$

respectively, with $Y_t^* = y^* e^{(\delta - r - \frac{1}{2}\underline{\Upsilon}^2)t - (\Sigma^{-1}\nu^*)^\top dW_t^*}$.

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Conclusion

- To deal with three features: (i) the return ambiguity, (ii) the optimal stopping time; (iii) a general class of utility requiring a nonnegative consumption and bequest, propose a dual optimal G-stopping approach.
- We establish the dual conjugate relationship between the primal problem with return ambiguity and the dual optimal stopping problem with risk ambiguity.
- Based on the analytical results, we derive the following features of the robust investment strategy: sparse investment in risky assets and least exposure to the ambiguity risk.
- Future extensions: Unbounded random parameters, Ambiguity in Mortality rate, Habit formation and ambiguity, Numerical algorithm for robust stopping time.

Thank You!