Asset Pricing with Liquidity Risk

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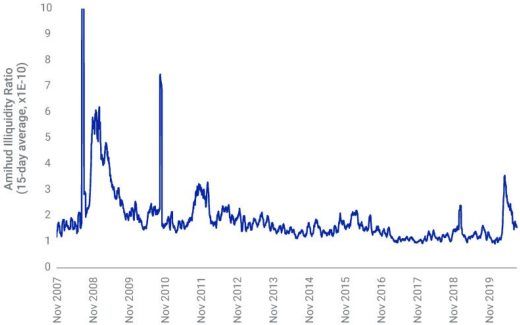
Introduction

Liquidity Risk and Asset Prices

- How do asset prices depend on *liquidity*?
  - "Liquidity discounts" for assets that are difficult to trade?
- Market liquidity is not constant but fluctuates over time.
- Additional discount for bearing this *liquidity risk*?
- Analysis of the underlying mechanisms requires *equlibrium* asset pricing models.
  - Prices not exogenous, but determined by market clearing.
  - Allows to study dependence on (exogenous) liquidity.
  - Major challenge: tractability.

Introduction
Liquidity Risk

- (Il)liquidity proxied by Amihud's 'ILLIQ":
  - Average of absolute price changes over trading volume.



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Introduction

Static Models

- Impact of transaction costs easy to analyze in *static models*.
  - One period models or prespecified holding periods as in Amihud/Mendelson '86.
  - Returns *net* of costs need to compensate for risk.
  - Illiquidity discounts determined by ratio (expected) trading costs and holding times.
- Can incorporate liquidity risk (Acharya/Pedersen '05).
  - Stochastic costs are just another risk factor in "liquidity adjusted CAPM".
  - Always compensated if systematic across assets but uncorrelated with other risk factors, for example.
- But this analysis crucially hinges on fixed holding periods!

Introduction

Dynamic Models

- Key insight from partial equilibrium (Constantinides '86):
  - *Investors accommodate large transaction costs by drastically reducing the frequency and volume of trade*.
- With liquidity risk?
  - Trading slows down in times of low liquidity, speeds up in times of high liquidity.
  - Compare, e.g., Kallsen/M-K '15, Moreau/M-K/Soner '17.
- How do asset prices reflect liquidity and liquidity risk in such dynamic models?
- Interplay of liquidity, trading needs, and asset prices?

Introduction

This Paper

- Starting point: Lo/Mamaysky/Wang '04.
  - Dynamic model for equilibrium asset pricing with (fixed) transaction costs.
  - Clever modelling choices allow to determine equilibrium price via a *one-dimensional* free boundary problem.
  - Leads to explicit formulas in the small-cost limit.
  - But cost is constant over time. No liquidity risk.
- We add liquidity risk to this model.
- Also allow trading needs to fluctuate over time, correlated with liquidity risk.
- We consider costs proportional to traded volumes as in Acharya/Pedersen '05.
  - Results are robust with respect to this specification.

Introduction

What we do

- Very generally, equilibrium asset prices and optimal trading strategies correspond to FBSDEs.
  - Optimal positions evolve forward from initial holdings.
  - Optimal adjustments and corresponding certainty equivalents determined by "backward induction".
- Systems are multidimensional and fully coupled.
- General wellposedness results still out of reach.
- But, surprisingly explicit results obtain in the small cost limit.
  - Equilibrium price decouples asymptotically.
  - Other frictional state process "average out" at the leading order, leading to a time-average of frictionless quantities only.
  - Leads to explicit formulas in concrete specifications.

The Model of Lo/Mamaysky/Wang '04

Frictionless Risk-Sharing Economy

- Economy with two assets:
  - Risk-free asset with exogenous interest rate $r > 0$.
  - Risky asset that pays cumulative dividends:

$$dD_t = a_D dt + \sigma_D dB_t$$

- Two agents trade to maximize exponential utility from consumption:

$$E\left[\int_0^\infty e^{-\delta t} U(c_t) ds\right], \quad \text{where } U(x) = -e^{-\gamma x}$$

- Trading motive: share risk from random endowments

$$dN_t^1 = X_t^1 dB_t \qquad dN_t^2 = X_t^2 dB_t$$

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The Model of Lo/Mamaysky/Wang '04

Frictionless Equilibrium

- Equilibrium prices determined by matching agents' demand to exogenous supply $2\bar{\theta}$.
- For tractability: $X_t^1 = \sigma_X B_t^\perp = -X_t^2$.
  - Zero *aggregate* endowment. Endowment volatilities driven by second Brownian motion uncorrelated with dividend shocks.
- Frictionless equilibrium price is constant: $\bar{P} = \frac{a_D}{r} - \bar{\theta}\gamma\sigma_D^2$.
  - Present value of future dividends discounted for dividend risk.
  - Frictionless holdings $\bar{\theta}_t^i = \bar{\theta} - \frac{X_t^i}{\sigma_D}$ perfectly hedge endowment shocks.
  - Endowment risks are shared perfectly and therefore not priced
  - Trading needs modulated by volatilities of endowment volatilities X_t^n. Constant over time here.
- How do these results change with transaction costs?

The Model of Lo/Mamaysky/Wang '04

Economy with Proportional Transaction Costs

- As in Acharya and Pedersen (2005):
  - Trading θ_t shares incurs proportional costs $\lambda|\theta_t|$.
  - Transaction tax or fee paid to trading platform.
- Equilibrium price is still constant, determined by free-boundary problem.
- For small transaction costs λ: explicit approximation

$$P_t = \bar{P}_t - \bar{\theta}\gamma\sigma_D^2 \left(\frac{2}{3} r\gamma \frac{\sigma_X^4}{\sigma_D^2}\right)^{1/3} \lambda^{2/3} + O(\lambda)$$

  - Liquidity discount of order $O(\lambda^{2/3})$ – unlike for homogenous quadratic preferences (Herdegen/M-K/Possamai '21).
  - Comparative statics analogous to result for fixed costs or quadratic costs up to change of powers and constants.

Models with Liquidity Risk

Lo/Mamaysky/Wang with Liquidity Risk?

- Keep dividend process

$$dD_t = a_D dt + \sigma_D dB_t$$

  and random endowments unchanged:

$$dN_t^1 = X_t^1 dB_t \qquad dN_t^2 = X_t^2 dB_t,$$

  where $X_t^1 = \sigma_X B_t^{\perp} = -X_t^2$ for $B_t^{\perp}$ uncorrelated with B_t.
- But now suppose trading costs are *stochastic*:
  - Trading θ_t shares still incurs proportional costs $\lambda_t |\theta_t|$.
  - But now $\lambda_t = \lambda \Lambda_t$, where scaling parameter λ describes *level* of liquidity, stochastic process Λ_t models its dynamics.
  - For tractability: λ_t also uncorrelated with dividend shocks.

Models with Liquidity Risk

Lo/Mamaysky/Wang with Liquidity Risk?

- For a given asset price, perturbation argument leads to FBSDE for optimal holdings.
  - Singular version (Shi '20) of the regular perturbations for quadratic costs.
  - Leads to reflection: controls only act when marginal valuations exceed thresholds.
  - Unlike for quadratic preferences, optimal strategies are also coupled to certainty equivalents.
- Corresponding equilibrium price solves another BSDE.
  - For a finite time horizon, liquidating dividend is the terminal condition.
  - Replaced by appropriate transversality condition for stationary infinite-horizon models.
- In equilibrium, only need to track one agent's strategy due to market clearing, but both certainty equivalents.

Models with Liquidity Risk

FBSDE System

- In summary, we obtain a system of four coupled BSDEs:
  - Each agent's certainty equivalent.
  - The agents' marginal valuation of the risky asset.
  - The equilibrium price.
- Also coupled to a singular forward process – the agents' risky holdings.
- System is nonlinear and fully coupled.
- With liquidity risk: no linear-quadratic special cases, even for quadratic costs and preferences.
- General wellposedness results unclear.
- Small-cost asymptotics also initially look daunting..

Models with Liquidity Risk

FBSDE Asymptotics

- But: system can be decoupled in the small-cost limit.
- Optimal trading strategy with costs can be approximated by its counterpart for the *frictionless* equilibrium price.
  - Known in closed form (Soner/Touzi '13, Martin '14, Kallsen/M-K '14).
  - Same approximation also works for certainty equivalents.
- Can then plug these expressions into the equation for the equilibrium price.
- Then, linearize and integrate out "fast variables" against their stationary distributions.
- Leads to an autonomous *linear* BSDE for the leading-order adjustment of the equilibrium due to small transaction costs!

Models with Liquidity Risk

Equilibrium Price

- Equilibrium price for small transaction costs:

$$P_t = \bar{P} - \bar{\theta}\gamma\sigma_D^2 \left(\frac{2}{3} r\gamma \frac{\sigma_X^4}{\sigma_D^2}\right)^{1/3} \int_t^{\infty} re^{-r(t-s)} E_t[\lambda_s^{1/2}] ds$$

- Illiquidity discount *lower than* if trading costs were known.
  - Implied by Jensen's inequality – for any form of liquidity risk.
- Initially puzzling – intuition?
  - In one-shot models like Acharya/Pedersen '05, extra uncorrelated liquidity risk is always bad.
  - However, in dynamic, agents can "time the market" – trade more in time of high liquidity.
  - With constant trading needs, randomly fluctuating costs are then preferable to their average level!

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Models with Liquidity Risk

Stochastic Trading Costs and Trading Needs?

- Dividend process still unchanged:

$$dD_t = a_D dt + \sigma_D dB_t$$

- But now make trading motives stochastic as well:

$$dN_t^1 = X_t^1 dB_t \qquad dN_t^2 = X_t^2 dB_t,$$

  where endowment volatilities $X_t^1 = -X_t^2$ have *general* Itô dynamics driven by orthogonal Brownian motion $B_t^{\perp}$.
- Volatilities can be correlated with trading costs λ_t.
- Two challenges:
  - Can we still solve for the corresponding frictionless equilibrium?
  - Leading-order adjustments due to small transaction costs?

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Models with Liquidity Risk

General Frictionless Equilibrium

- With frictionless trading, perfect risk sharing is still possible.
- Constant equilibrium price $\bar{P}$ still only depends on unchanged dividends.
- Corresponding trading strategies still hedge all endowment risk – affine functions of the state process X_t.
- Frictionless certainty equivalents and their volatilities $(\bar{Y}, \bar{Z})$ solve quadratic BSDE.
  - Lo/Mamaysky/Wang nested as a special case.
  - Solution is still explicit for more general affine state dynamics.
- With the frictionless equilibrium at hand – what can we say about the leading order adjustment due to small trading costs?

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Models with Liquidity Risk

Impact of Small Transaction Costs

- Same approximation arguments still go through.
- Equilibrium price with small transaction costs still solves autonomous linear BSDE.
- Coefficients still only depend on the *frictionless* equilibrium
- But now also through the volatilities $\bar{Z}_t^1$, $\bar{Z}_t^2$ of the frictionless certainty equivalents:

$$P_t = \bar{P}_t + \frac{\gamma}{2}\tilde{E}_t\left[\int_t^T e^{-r(s-t)}\left(\frac{2}{3}r\gamma\sigma_D\sigma_s^X\right)^{1/3}\lambda_s^{2/3}(\bar{Z}_s^1 - \bar{Z}_s^2)ds\right]$$

for the measure $\tilde{\mathbb{P}}$ with density $\mathcal{E}(\int_0^\cdot -\frac{\gamma_1}{2}(\bar{Z}_t^1 + \bar{Z}_t^2)dB_t^\perp)$.

Models with Liquidity Risk

Examples

- To make explicit computations possible:
  - Need to know volatilities $\bar{Z}_t^1$, $\bar{Z}_t^2$ of frictionless certainty equivalents in closed form.
  - These processes and the trading cost need to remain tractable after the change of measure to compute expectation.
- Simplest case: Lo/Mamaysky/Wang.
  - $\bar{Z}^1$, $\bar{Z}^2$ are constant.
  - No change of measure.
  - Recover previous result where liquidity risk always reduces illiquidity discounts.
- Tractable models with changing trading needs?

Models with Liquidity Risk

Examples ct'd

- Replace constant volatility σ_X of frictionless hedging strategy X_t by a CIR process V_t.
  - Inspired frictionless (but incomplete) equilibrium studied by Munk/Larsen '13.
- Frictionless certainty equivalents are linear in X_t and V_t, volatility is linear in $\sqrt{V_t}$.
  - Coefficients can be computed in closed form by solving quadratic equations.
- V_t remains CIR process (with adjusted coefficients) after change of measure.
- Model transaction costs as power of the same state process.
  - Unlike the AR(1) process of Acharya/Pedersen '05 this guarantees positive trading costs.

Models with Liquidity Risk

Examples ct'd

- Illiquidity discounts determined by (fractional) powers of CIR processes.
  - Lo/Mamaysky/Wang model is recovered in the limiting case of fast mean-reversion.
- Simplest case: $\lambda(V_t) \propto \sqrt{V_t}$.
- Then, illiquidity discount can be computed from the (closed-form) expectation of the CIR process.
- After the dust settles, this leads to a complicated but fully explicit formula for the illiquidity discount.
  - Comparative statics follow from somewhat tedious but straightforward computations.
- Impact of liquidity risk?

Models with Liquidity Risk

Examples ct'd

- In this model, illiquidity discount $\bar{P}_t - P_t$ is always *bigger* than without liquidity risk.
  - Follows from Jensen's inequality and the comparison theorem for one-dimensional diffusions.
- What is different compared to Lo/Mamaysky/Wang?
  - Trading costs and trading needs are now *perfectly* correlated.
  - Trading costs high when hedging against endowment risk is most needed.
  - Intertemporal substitution of trades no longer as efficient.
- Combination of both models: investors demand a premium for liquidity risks if trading needs are *sufficiently* correlated.

Summary and Outlook

Main Punchlines

Study fills an important gap in the literature on transaction costs:

- Lo et al. (2004): dynamic general equilibrium model, but with constant transaction costs.
  - Cannot assess how risk of liquidity shocks affects illiquidity discounts.
- Acharya and Pedersen (2005): equilibrium asset pricing with liquidity risk
  - Setting up and liquidating portfolios is the only motive to trade.
  - Focus primarily on the cross-sectional distribution of liquidity
  - One-period investors in overlapping generation model.
  - Cannot capture how time-varying dimension of liquidity is reflected with dynamic trading and intertemporal hedging.

Summary and Outlook

Next Steps

- Interpretation of general price adjustment?
  - Change of measure finally identified, but what about the difference of certainty equivalent volatilities?
- Wellposedness of FBSDE systems?
  - Partial results for quadratic costs and preferences (Herdegen/M-K/Possamai '21).
  - More generally?
- Numerical solution of the (exact) FBSDE system?
- Magnitude of the effects for reasonable parameters?
  - Model is tractable enough to produce testable implications.
  - But proxying both the cross section and time series of liquidity are a challenge ($\rightsquigarrow$ in progress).
- Endogenize the interest rate as well?

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