Asset Pricing with Liquidity Risk

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Introduction
Liquidity Risk and Asset Prices

- How do asset prices depend on *liquidity*?
  - “Liquidity discounts” for assets that are difficult to trade?
- Market liquidity is not constant but fluctuates over time.
- Additional discount for bearing this *liquidity risk*?
- Analysis of the underlying mechanisms requires *equilibrium* asset pricing models.
  - Prices not exogenous, but determined by market clearing.
  - Allows to study dependence on (exogenous) liquidity.
  - Major challenge: tractability.
Introduction
Liquidity Risk

» (Il)liquidity proxied by Amihud’s ‘ILLIQ’:
  » Average of absolute price changes over trading volume.
Introduction
Static Models

- Impact of transaction costs easy to analyze in *static models*.
  - One period models or prespecified holding periods as in Amihud/Mendelson ‘86.
  - Returns *net* of costs need to compensate for risk.
  - Illiquidity discounts determined by ratio (expected) trading costs and holding times.
- Can incorporate liquidity risk (Acharya/Pedersen ‘05).
  - Stochastic costs are just another risk factor in “liquidity adjusted CAPM”.
  - Always compensated if systematic across assets but uncorrelated with other risk factors, for example.
- But this analysis critically hinges on fixed holding periods!
Introduction
Dynamic Models

- Key insight from partial equilibrium (Constantinides ‘86):
  - *Investors accommodate large transaction costs by drastically reducing the frequency and volume of trade.*

- With liquidity risk?
  - Trading slows down in times of low liquidity, speeds up in times of high liquidity.
  - Compare, e.g., Kallsen/M-K ‘15, Moreau/M-K/Soner ‘17.

- How do asset prices reflect liquidity and liquidity risk in such dynamic models?

- Interplay of liquidity, trading needs, and asset prices?
Introduction

This Paper

- Starting point: Lo/Mamaysky/Wang ‘04.
  - Dynamic model for equilibrium asset pricing with (fixed) transaction costs.
  - Clever modelling choices allow to determine equilibrium price via a *one-dimensional* free boundary problem.
  - Leads to explicit formulas in the small-cost limit.
  - But cost is constant over time. No liquidity risk.

- We add liquidity risk to this model.

- Also allow trading needs to fluctuate over time, correlated with liquidity risk.

- We consider costs proportional to traded volumes as in Acharya/Pedersen ‘05.
  - Results are robust with respect to this specification.
Introduction

What we do

- Very generally, equilibrium asset prices and optimal trading strategies correspond to FBSDEs.
  - Optimal positions evolve forward from initial holdings.
  - Optimal adjustments and corresponding certainty equivalents determined by “backward induction”.
- Systems are multidimensional and fully coupled.
- General wellposedness results still out of reach.
- But, surprisingly explicit results obtain in the small cost limit.
  - Equilibrium price decouples asymptotically.
  - Other frictional state process “average out” at the leading order, leading to a time-average of frictionless quantities only.
  - Leads to explicit formulas in concrete specifications.
The Model of Lo/Mamaysky/Wang ’04
Frictionless Risk-Sharing Economy

- Economy with two assets:
  - Risk-free asset with exogenous interest rate $r > 0$.
  - Risky asset that pays cumulative dividends:
    \[ dD_t = a_D dt + \sigma_D dB_t \]

- Two agents trade to maximize exponential utility from consumption:
  \[ E \left[ \int_0^\infty e^{-\delta t} U(c_t) ds \right], \quad \text{where} \quad U(x) = -e^{-\gamma x} \]

- Trading motive: share risk from random endowments
  \[ dN^1_t = X^1_t dB_t \quad dN^2_t = X^2_t dB_t \]
Equilibrium prices determined by matching agents’ demand to exogenous supply $2\bar{\theta}$.

For tractability: $X_t^1 = \sigma X B_t^\perp = -X_t^2$.

- Zero aggregate endowment. Endowment volatilities driven by second Brownian motion uncorrelated with dividend shocks.

Frictionless equilibrium price is constant: $\bar{P} = \frac{aD}{r} - \bar{\theta} \gamma \sigma_D^2$.

- Present value of future dividends discounted for dividend risk.
- Frictionless holdings $\bar{\theta}_t^i = \bar{\theta} - \frac{X_t^i}{\sigma_D}$ perfectly hedge endowment shocks.
- Endowment risks are shared perfectly and therefore not priced.
- Trading needs modulated by volatilities of endowment volatilities $X_t^n$. Constant over time here.

How do these results change with transaction costs?
As in Acharya and Pedersen (2005):
- Trading $\theta_t$ shares incurs proportional costs $\lambda|\theta_t|$.
- Transaction tax or fee paid to trading platform.

Equilibrium price is still constant, determined by free-boundary problem.

For small transaction costs $\lambda$: explicit approximation

$$P_t = \bar{P}_t - \bar{\theta}\gamma\sigma_D^2 \left( \frac{2}{3} r \gamma \frac{\sigma_X^4}{\sigma_D^2} \right)^{1/3} \lambda^{2/3} + O(\lambda)$$

- Liquidity discount of order $O(\lambda^{2/3})$ – unlike for homogenous quadratic preferences (Herdegen/M-K/Possamai ‘21).
- Comparative statics analogous to result for fixed costs or quadratic costs up to change of powers and constants.
Models with Liquidity Risk
Lo/Mamaysky/Wang with Liquidity Risk?

- Keep dividend process

\[ dD_t = a_D dt + \sigma_D dB_t \]

and random endowments unchanged:

\[ dN^1_t = X^1_t dB_t \quad dN^2_t = X^2_t dB_t, \]

where \( X^1_t = \sigma_X B^\perp_t = -X^2_t \) for \( B^\perp_t \) uncorrelated with \( B_t \).

- But now suppose trading costs are stochastic:
  - Trading \( \theta_t \) shares still incurs proportional costs \( \lambda_t |\theta_t| \).
  - But now \( \lambda_t = \lambda \Lambda_t \), where scaling parameter \( \lambda \) describes level of liquidity, stochastic process \( \Lambda_t \) models its dynamics.
  - For tractability: \( \lambda_t \) also uncorrelated with dividend shocks.
Models with Liquidity Risk
Lo/Mamaysky/Wang with Liquidity Risk?

- For a given asset price, perturbation argument leads to FBSDE for optimal holdings.
  - Singular version (Shi ‘20) of the regular perturbations for quadratic costs.
  - Leads to reflection: controls only act when marginal valuations exceed thresholds.
  - Unlike for quadratic preferences, optimal strategies are also coupled to certainty equivalents.

- Corresponding equilibrium price solves another BSDE.
  - For a finite time horizon, liquidating dividend is the terminal condition.
  - Replaced by appropriate transversality condition for stationary infinite-horizon models.

- In equilibrium, only need to track one agent’s strategy due to market clearing, but both certainty equivalents.
Models with Liquidity Risk
FBSDE System

- In summary, we obtain a system of four coupled BSDEs:
  - Each agent’s certainty equivalent.
  - The agents’ marginal valuation of the risky asset.
  - The equilibrium price.
- Also coupled to a singular forward process – the agents’ risky holdings.
- System is nonlinear and fully coupled.
- With liquidity risk: no linear-quadratic special cases, even for quadratic costs and preferences.
- General wellposedness results unclear.
- Small-cost asymptotics also initially look daunting.
But: system can be decoupled in the small-cost limit.

Optimal trading strategy with costs can be approximated by its counterpart for the *frictionless* equilibrium price.

- Known in closed form (Soner/Touzi ‘13, Martin ‘14, Kallsen/M-K ‘14).
- Same approximation also works for certainty equivalents.

Can then plug these expressions into the equation for the equilibrium price.

Then, linearize and integrate out “fast variables” against their stationary distributions.

Leads to an autonomous *linear* BSDE for the leading-order adjustment of the equilibrium due to small transaction costs!
Equilibrium price for small transaction costs:

\[ P_t = \bar{P} - \bar{\theta} \gamma \sigma_D^2 \left( \frac{2}{3} r \gamma \frac{\sigma_X^4}{\sigma_D^2} \right)^{1/3} \int_t^\infty re^{-r(t-s)} E_t[\lambda_s^{1/2}] ds \]

Illiquidity discount lower than if trading costs were known.
- Implied by Jensen’s inequality – for any form of liquidity risk.
- Initially puzzling – intuition?
  - In one-shot models like Acharya/Pedersen ‘05, extra uncorrelated liquidity risk is always bad.
  - However, in dynamic, agents can “time the market” – trade more in time of high liquidity.
  - With constant trading needs, randomly fluctuating costs are then preferable to their average level!
Models with Liquidity Risk
Stochastic Trading Costs and Trading Needs?

- Dividend process still unchanged:
  \[ dD_t = a_D dt + \sigma_D dB_t \]

- But now make trading motives stochastic as well:
  \[ dN_1^t = X_1^t dB_t \quad dN_2^t = X_2^t dB_t, \]
  where endowment volatilities \( X_1^t = -X_2^t \) have general Itô dynamics driven by orthogonal Brownian motion \( B_t^\perp \).

- Volatilities can be correlated with trading costs \( \lambda_t \).

- Two challenges:
  - Can we still solve for the corresponding frictionless equilibrium?
  - Leading-order adjustments due to small transaction costs?
Models with Liquidity Risk

General Frictionless Equilibrium

- With frictionless trading, perfect risk sharing is still possible.
- Constant equilibrium price $\bar{P}$ still only depends on unchanged dividends.
- Corresponding trading strategies still hedge all endowment risk – affine functions of the state process $X_t$.
- Frictionless certainty equivalents and their volatilities ($\bar{Y}, \bar{Z}$) solve quadratic BSDE.
  - Lo/Mamaysky/Wang nested as a special case.
  - Solution is still explicit for more general affine state dynamics.
- With the frictionless equilibrium at hand – what can we say about the leading order adjustment due to small trading costs?
Models with Liquidity Risk
Impact of Small Transaction Costs

- Same approximation arguments still go through.
- Equilibrium price with small transaction costs still solves autonomous linear BSDE.
- Coefficients still only depend on the frictionless equilibrium.
- But now also through the volatilities $\bar{Z}_1^t$, $\bar{Z}_2^t$ of the frictionless certainty equivalents:

$$P_t = \bar{P}_t + \frac{\gamma}{2} \tilde{E}_t \left[ \int_t^T e^{-r(s-t)} \left( \frac{2}{3} r \gamma \sigma D \sigma_s X \right)^{1/3} \lambda_s^{2/3} (\bar{Z}_s^1 - \bar{Z}_s^2) ds \right]$$

for the measure $\tilde{\mathbb{P}}$ with density $\mathcal{E}(\int_0^t - \frac{\gamma_1}{2} (\bar{Z}_t^1 + \bar{Z}_t^2) dB_t^\perp)$. 
Models with Liquidity Risk

Examples

▶ To make explicit computations possible:
  ▶ Need to know volatilities \( \bar{Z}_t^1, \bar{Z}_t^2 \) of frictionless certainty equivalents in closed form.
  ▶ These processes and the trading cost need to remain tractable after the change of measure to compute expectation.

▶ Simplest case: Lo/Mamaysky/Wang.
  ▶ \( \bar{Z}^1, \bar{Z}^2 \) are constant.
  ▶ No change of measure.
  ▶ Recover previous result where liquidity risk always reduces illiquidity discounts.

▶ Tractable models with changing trading needs?
Models with Liquidity Risk

Examples ct’d

- Replace constant volatility $\sigma_X$ of frictionless hedging strategy $X_t$ by a CIR process $V_t$.
  - Inspired frictionless (but incomplete) equilibrium studied by Munk/Larsen ‘13.

- Frictionless certainty equivalents are linear in $X_t$ and $V_t$, volatility is linear in $\sqrt{V_t}$.
  - Coefficients can be computed in closed form by solving quadratic equations.

- $V_t$ remains CIR process (with adjusted coefficients) after change of measure.

- Model transaction costs as power of the same state process.
  - Unlike the AR(1) process of Acharya/Pedersen ‘05 this guarantees positive trading costs.
Models with Liquidity Risk
Examples ct’d

- Illiquidity discounts determined by (fractional) powers of CIR processes.
  - Lo/Mamaysky/Wang model is recovered in the limiting case of fast mean-reversion.

- Simplest case: \( \lambda(V_t) \propto \sqrt{V_t} \).

- Then, illiquidity discount can be computed from the (closed-form) expectation of the CIR process.

- After the dust settles, this leads to a complicated but fully explicit formula for the illiquidity discount.
  - Comparative statics follow from somewhat tedious but straightforward computations.

- Impact of liquidity risk?
In this model, illiquidity discount $\bar{P}_t - P_t$ is always bigger than without liquidity risk.

- Follows from Jensen’s inequality and the comparison theorem for one-dimensional diffusions.

What is different compared to Lo/Mamaysky/Wang?

- Trading costs and trading needs are now perfectly correlated.
- Trading costs high when hedging against endowment risk is most needed.
- Intertemporal substitution of trades no longer as efficient.

Combination of both models: investors demand a premium for liquidity risks if trading needs are sufficiently correlated.
Study fills an important gap in the literature on transaction costs:

- Lo et al. (2004): dynamic general equilibrium model, but with constant transaction costs.
  - Cannot assess how risk of liquidity shocks affects illiquidity discounts.
- Acharya and Pedersen (2005): equilibrium asset pricing with liquidity risk
  - Setting up and liquidating portfolios is the only motive to trade.
  - Focus primarily on the cross-sectional distribution of liquidity
  - One-period investors in overlapping generation model.
  - Cannot capture how time-varying dimension of liquidity is reflected with dynamic trading and intertemporal hedging.
Summary and Outlook

Next Steps

- Interpretation of general price adjustment?
  - Change of measure finally identified, but what about the difference of certainty equivalent volatilities?

- Wellposedness of FBSDE systems?
  - Partial results for quadratic costs and preferences (Herdegen/M-K/Possamai '21).
  - More generally?

- Numerical solution of the (exact) FBSDE system?

- Magnitude of the effects for reasonable parameters?
  - Model is tractable enough to produce testable implications.
  - But proxying both the cross section and time series of liquidity are a challenge (⇝ in progress).

- Endogenize the interest rate as well?