UNIVERSITÀ DEGLI STUDI DI MILANO
DIPARTIMENTO DI MATEMATICA
"FEDERIGO ENRIQUES"

A unifying approach to viability and arbitrage

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This talk is mainly based on a joint work with:

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Viability and arbitrage under Knightian Uncertainty.
Econometrica, 2021.

- Introduction
- Viability in the classical literature
- The model
- The main results
  - The statements and idea of the proofs;
- Efficient Market Hypothesis revisited
- Conclusions

Introduction

Volatility uncertainty is one of the most prominent example which calls for a change of paradigm in the classical approach to mathematical finance.

- [...] evidence suggests that relevant volatilities follow complicated dynamics, [...] however, one might question whether it is plausible to assume that agents become completely confident in any particular law of motion.
- We are interested in situations where realized past volatility may not be a reliable predictor of volatility in the future $\rightsquigarrow$ perceived ambiguity.
- At a technical level, the analysis requires a significant departure from existing continuous-time modeling because ambiguous volatility cannot be modeled within a probability space framework.
- The natural question is whether and in what form the cornerstones of received asset pricing theory extend to a framework with ambiguous volatility.

Epstein & Ji, Ambiguous Volatility and Asset Pricing in Continuous Time. *The Review of Financial Studies*, 2013.

These considerations generated a vast literature on fundamental questions of mathematical finance under model uncertainty.

- Quasi-sure FTAP & duality. Bouchard & Nutz (2015) + Biagini & Kardaras (2017), Bayraktar & Zhang (2016), Blanchard & Carassus (2018).
- Pointwise FTAP & duality. Riedel (2015), Acciaio, Beiglböck, Penkner, Schachermayer (2016), B., Frittelli, Maggis (2016a, 2016b), + Hu & Obloj (2019), Bartl, Cheridito, Kupper & Tangpi (2017).
- Game theoretical duality. Vovk (2011-2015), Beiglböck, Cox, Huesmann, Perkowski, Pröemel (2017).
- Martingale optimal transport duality. Beiglböck, Henry-Labordère, Penkner (2013), Galichon, Henry-Labordère, Touzi (2014), Cheridito, Kiiski, Pröemel & Soner (2020).

Why do we care?

These type of results pose the ground for studying any other research question in Mathematical Finance.

- They justify the use of martingale modeling (of some type) as they represent arbitrage free markets (of some type).
- They justify a certain output if it is compatible with no arbitrage bounds (of some type).
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- etc...

However, economists typically think in terms of agents, preferences and equilibrium.

Q: Is this no arbitrage approach economically sound?

Classical viability

- Harrison & Kreps, Martingales and Arbitrage in Multiperiod Securities Markets *Journal of Economic Theory*, 1979.
- Kreps, Arbitrage and Equilibrium in Economies With Infinitely Many Commodities. *Journal of Mathematical Economics*, 1981.

Let $\mathcal{H}$ be a space of random variables representing a set of financial claims of interest, e.g., $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Suppose we are given a price system (M, π), i.e.,

- a linear space $M \subset \mathcal{H}$ of marketed claims,
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Q: Does this follow from some economic principle?

(M, π) is defined to be viable if there exist $r^* \in \mathbb{R}$, a preference relation $\preceq$ and $m^* \in M$ such that

- budget constraint is fulfilled: $r^* + \pi(m^*) \leq 0$.
- optimality: $(r^*, m^*) \succeq (r, m)$ for all $r \in \mathbb{R}, m \in M$ with $r + \pi(m) \leq 0$

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Under some conditions on $\preceq$ (conceivable agents) we have the following

Theorem (K78, HK79)

$$\textit{viability} \iff \pi \textit{ admits an extension to } \mathcal{H}$$

Consider the sets

$$A = \{(r, x) \succ (0, 0)\}, \qquad B = \{r + \pi(m) \leq 0\}$$

- If we suppose that preferences are continuous and convex, both sets are convex and A is open;
- By viability they are disjoint: w.l.o.g. $(r^*, m^*) = (0, 0)$;
- Separating the two sets we construct a linear functional ψ that extends π and is strictly positive on positive claims.

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Theorem (HK79)

viability "$\Longleftrightarrow$" $\mathbb{P}$ *admits an equivalent martingale measure*

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- Absence of arbitrage and viability are not equivalent in that framework.
- In the probabilistic model of [HK79] viability and no arbitrage are assumed together in the definition.
- The model is on a general $\mathcal{H}$ but it requires that the linear functional ψ is strictly positive on a given cone K (in the case above $L^2_+(\Omega, \mathcal{F}, \mathbb{P}) \setminus \{0\}$).

Viability under Knightian Uncertainty

In general, strictly positive linear functionals do not exist. A trivial example is the following *pointwise framework*.

Example

Let $\Omega := [0,1]$, $\mathcal{H} := \mathcal{L}^0(\Omega, \mathcal{F})$, $\geq$ the pointwise order and

$$\{x \in \mathcal{H} : x \geq 0 \text{ and } x(\omega) > 0, \text{ for some } \omega \in \Omega\}$$

as the class of strictly positive elements.

It is well known that there exists no countably additive measure μ with $\mu(\{\omega\}) > 0$ for every $\omega \in \Omega$.

Example

Consider the classical Black-Scholes model, the stock price S satisfies $dS_t = \sigma S_t dB_t$ where B is a standard Brownian motion but there is uncertainty about the volatility and σ is any process in the set

$$\Sigma := \left\{ \sigma : [0,T] \to [\underline{\sigma}, \overline{\sigma}] \mid \sigma \text{ is adapted } \right\}.$$

Let $\mathbb{P}^\sigma$ be the distribution of the stock price process with volatility process σ. The class $\mathcal{P} := \{\mathbb{P}^\sigma\}$ contains mutually singular priors and is not dominated by a common prior.

The model

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- A general class of contingent claims $\mathcal{H}$. A probability space is not needed, e.g., $B_b(\Omega)$.

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- A class of net trades $\mathcal{I}$ available in the market at zero cost.
- A class of relevant (strictly positive) elements $\mathcal{R}$ which is a subset of $\{X > 0\}$.
- A class of agents with monotone, convex and lower semi-continuous preferences: $x_n \to x$ with $x_n \preceq y$ $\forall n \in \mathbb{N}$ implies $x \preceq y$.

The common order is a crucial component. It reflects the type of assumptions we are willing to make on the financial market.

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3. $\mathcal{P}$-q.s. order: $X \geq Y$ iff $\mathbb{P}(X \geq Y) = 1\ \forall \mathbb{P} \in \mathcal{P}$;
4. induced by preferences: Let $\{\preceq_a\}_{a \in \mathcal{A}}$ be a class of agents and define negligible claims as

$$\mathcal{Z} := \bigcap_{a \in \mathcal{A}} \{Z \in \mathcal{H} \ : \ X + Z \sim_a X \quad \forall X \in \mathcal{H}\}.$$

Let $X \geq Y$ iff $X(\omega) \geq Y(\omega) + Z(\omega)\ \forall \omega \in \Omega$ for some $Z \in \mathcal{Z}$.

The net trades represents the trading opportunities available at zero cost in the market. We assume it is a convex cone.

Example

1. One-period frictionless markets: $\mathcal{I} = \{H \cdot (S_T - S_0) \ : \ H \in \mathbb{R}^d\}$;

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Dolinsky & Soner, Martingale optimal transport and robust hedging in continuous time, *Probability Theory and Related Fields*, 2014.

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3. Markets under uncertainty: simple integrands or of bounded variation;

   Dolinsky & Soner, Martingale optimal transport and robust hedging in continuous time, *Probability Theory and Related Fields*, 2014.
4. Markets with frictions, e.g., no short sales.

In classical frameworks it is natural to take $\mathcal{R} = \{X > 0\}$. Under Knightian Uncertainty this may be a too large set.

Example

Pointwise order: $\{X > 0\}$ is the set of non-negative claims such that for at least one $\bar{\omega}$, $X(\bar{\omega}) > 0$. Alternatives:

- $\mathcal{R}_{open} := \left\{ R \in C_b(\Omega)_+ \ : \ \exists \bar{\omega} \in \Omega \text{ such that } R(\bar{\omega}) > 0 \right\}.$

  B., Frittelli, Maggis, Universal arbitrage aggregator in discrete-time markets under uncertainty. *Finance and Stochastics*, 2016.

- $\mathcal{R}_{++} = \left\{ R \in \mathcal{H} \ : \ R(\omega) > 0 \quad \forall \omega \in \Omega \right\}.$

  Acciaio, Beiglböck, Penkner, Schachermayer, A Model-Free Version of the Fundamental Theorem of Asset Pricing and the Super-Replication Theorem. *Mathematical Finance*, 2016.

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Example

Volatility uncertainty: $\{X > 0\}$ contains the indicators of any volatility process $\sigma \in [\underline{\sigma}, \overline{\sigma}]$. Alternative: For $\varepsilon > 0$, let

$$\Sigma_{\sigma,\varepsilon} := \{\tilde{\sigma} \in \Sigma \ : \ \|\sigma - \tilde{\sigma}\|_\infty \leq \varepsilon \ \}$$

and define relevant sets as:

$$\mathcal{R} = \Big\{R \in \mathcal{P}_+ : \inf_{\sigma \in \Sigma_{\sigma,\varepsilon}} \mathbb{E}_{\mathbb{P}^\sigma}[R] > 0 \text{ for some } \ \sigma \in \Sigma, \varepsilon > 0\Big\}.$$

The main results

A financial market is viable if there is a family of agents $\{\preceq_a\}_{a\in A} \subset \mathcal{A}$ such that:

- 0 is optimal for each agent $a \in A$, i.e.

$$\forall \ell \in \mathcal{I} \quad \ell \preceq_a 0,$$

- for every relevant claim $R \in \mathcal{R}$ there exists an agent $a \in A$ such that

$$0 \prec_a R\,.$$

i.e. an heterogeneous agents' economy in equilibrium vs a single representative agent economy of Harrison & Kreps.

Define arbitrage in a standard way: $\ell \in \mathcal{I}$ is an *arbitrage* if $\exists R \in \mathcal{R}$ with $\ell \geq R$ and a sequence of net trades $\{\ell_n\}$ is a *free lunch with vanishing risk* if $c_n + \ell_n \geq R$ with $c_n \downarrow 0$.

Theorem (B., Riedel, Soner)

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Comments:

- absence of arbitrage is the starting point of any reasonable analysis in mathematical finance. We provide an economics justification of such assumption.
- this results was not obtained in Harrison and Kreps.

Theorem (B., Riedel, Soner)

The following are equivalent:

1. *viability*
2. *there exists a sublinear expectation* $\mathcal{E} : \mathcal{H} \to \mathbb{R}$ *which satisfies:*
   - *monotonicity with respect to* $\leq$*;*
   - *sublinearity:* $\mathcal{E}(x + y) \leq \mathcal{E}(x) + \mathcal{E}(y)$ *for every* $x, y \in \mathcal{H}$*;*
   - *translation-invariance:* $\mathcal{E}(x + c) = \mathcal{E}(x) + c$ *for* c *constant;*
   - *"martingale property":* $\mathcal{E}(\ell) \leq 0$ *for every* $\ell \in \mathcal{I}$*;*
   - *"full support property":* $\mathcal{E}(R) > 0$ *for every* $R \in \mathcal{R}$*.*

Suppose $\mathcal{H}$ is the set of bounded measurable functions. Then, $\mathcal{H}' = ba(\Omega, \mathcal{F})$. Consider the super-hedging functional

$$\mathcal{D}(x) := \inf\{c : \exists \ell \subset \mathcal{I} \text{ s.t. } c + \ell \geq X\}.$$

One can show:

- NFLVR is equivalent to $\mathcal{D}(R) > 0$ for every $R \in \mathcal{R}$;
- $\mathcal{D}$ is $\|\cdot\|_\infty$-l.s.c. and proper;
- By Fenchel Moreau $\mathcal{D}(x) = \sup_{\varphi \in \mathcal{H}'} \left\{\varphi(x) - \mathcal{D}^*(\varphi)\right\}$ where

$$\mathcal{D}^*(\varphi) = \sup_{y \in \mathcal{H}} \left\{\varphi(y) - \mathcal{D}(y)\right\};$$

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One can show:

- $\mathcal{D}(x) = \sup_{\varphi \in dom(\mathcal{D}^*)} \ \varphi(x)$ where

$$dom(\mathcal{D}^*) = \{\varphi \in \mathcal{H}' : \mathcal{D}^*(\varphi) < \infty\};$$

- every $\varphi \in dom(\mathcal{D}^*)$ satisfies the martingale property;
- $\forall R \in \mathbb{R}$, there exists $\varphi \in dom(\mathcal{D}^*)$ s.t. $\varphi(R) > 0$.

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- Second, $\mathcal{D}$ can be represented by a class of martingale functionals $dom(\mathcal{D}^*)$ which is, in particular, non-empty.
- For each $\varphi \in dom(\mathcal{D}^*)$, define $\preceq_\varphi$ by,

$$X \preceq_\varphi Y, \quad \Leftrightarrow \quad \varphi(X) \leq \varphi(Y).$$

One verifies that $\preceq_\varphi \in \mathcal{A}$. Moreover, $\varphi(\ell) \leq \varphi(0) = 0$ for any $\ell \in \mathcal{I}$ implies that $\ell^*_\varphi = 0$ is optimal.

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- Finally, for any $R \in \mathcal{R}$, there exists $\varphi \in dom(\mathcal{D}^*)$ such that $\varphi(R) > 0$.

Viability $\Rightarrow$ absence of arbitrage.

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- By optimality of the zero trade, $\ell_n \preceq_a 0$, and we get $-e_n + R^* \preceq_a 0$. By lower semi–continuity of $\preceq_a$, we conclude that $R^* \preceq_a 0$, a contradiction.

Efficient Market Hypothesis

In general, the Efficient Market Hypothesis usually refers to the following statement:

All the available information is reflected properly in current asset prices.

Fama, Efficient capital markets: A review of theory and empirical work. *The Journal of Finance*, 1970.

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Different interpretations: normality of log-returns, independence of increments, martingale property of asset prices, etc.

The form of the EMH depends on the hypothesis one is willing to take on the common order. Suppose we work in a frictionless market with price process S.

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3. $X \geq Y$ iff $\mathbb{E}_{\mathbb{P}}[X] \geq \mathbb{E}_{\mathbb{P}}[Y]$ $\forall \mathbb{P} \in \mathcal{P} \Rightarrow \exists \mathcal{Q} \subset \mathcal{P}$ such that S is a symmetric $\mathcal{Q}$-martingale.

The form of the EMH depends on the hypothesis one is willing to take on the common order. Suppose we work in a frictionless market with price process S.

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Conclusions

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- We presented a general framework which is able to include classical models of risk as well as models of Knightian Uncertainty;
- With the help of an heterogeneous agents economy we are able to prove the equivalence between absence of arbitrage, viability and existence of pricing functionals, which are in general sublinear;
- We showed how to recover different versions of the Efficient Market Hypothesis according to the hypothesis we assume on the common order.

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Thank you for your kind attention!

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