Term structure modelling with overnight rates beyond stochastic continuity

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The LIBOR reform: a bit of history

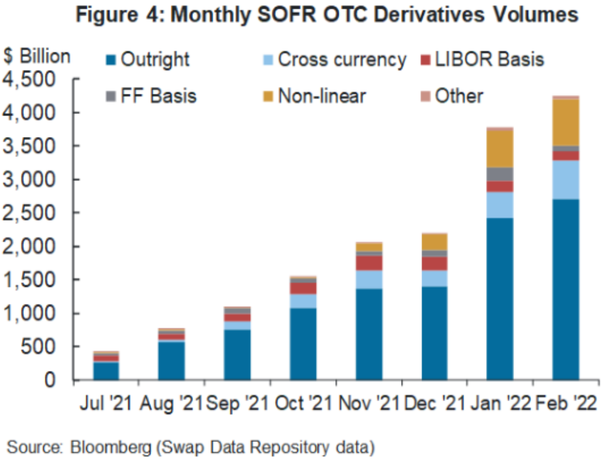
- London Interbank Offered Rate (LIBOR), computed as the trimmed average of rates reported by a panel of banks, for five currencies (CHF, EUR, GBP, JPY, USD) and seven tenors (1D, 1W, 1M, 2M, 3M, 6M, 1Y).
- LIBOR was launched in 1986 and widely adopted as benchmark rate.
- Starting from 2010, the volume of uncollateralized loans in the interbank market shrinked significantly, due to counterparty risk and other causes.
- 2012: evidence of LIBOR manipulation by several major banks.
- July 2017: *The future of LIBOR* speech by Andrew Bailey (CEO of FCA): LIBOR has no future! It will be discontinued after 2021.
- Transition towards transaction-based overnight rates as benchmark rates. ARRC, June 2017: Secured Overnight Funding Rate (SOFR) in the US.
- FCA, March 2021: cessation of LIBOR on 31/12/2021.
- May 2021: *Life after LIBOR* speech by Andrew Bailey:

  > *"transition to the most robust overnight rates, underpinned by deep underlying markets, will support a stronger more transparent financial system and ultimately benefit all market participants".*

Alternative risk-free rates

- SOFR uses data from overnight Treasury repo trades to calculate a rate published at 8:00 a.m. (NY time) by the Federal Reserve Bank.
- The scale of SOFR's underlying transaction pool is massive.
  The transaction volume underlying SOFR is around 1 USD trillion daily.
- Various filters, trims, and inclusion rules are applied to these data sources and a transaction weighted median becomes the benchmark rate.
- Sterling Overnight Index Average (SONIA) in the UK;
- Tokyo Overnight Average Rate (TONA) in Japan;
- Swiss Average Rate Overnight (SARON) in Switzerland;
- Euro Short Term Rate (€STR) in the EU.

Alternative risk-free rates

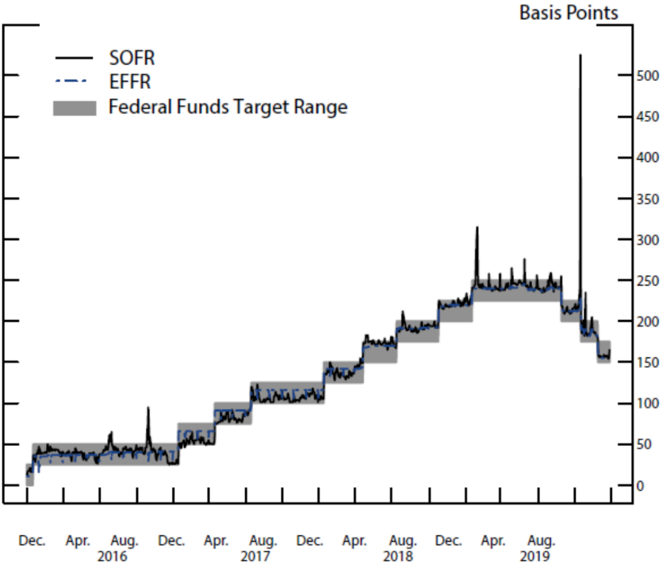


Alternative risk-free rates

- Alternative risk-free rates (RFRs) are nearly risk-free overnight rates;
- Being risk-free, RFRs should reflect the current level of policy rates:
  as documented by Backwell and Hayes (2021), most of the variation in SONIA over the years 2016-2020 occurs in correspondence to the meeting dates of the Monetary Policy Committee of the Bank of England.
  The meeting dates typically follow a predetermined calendar.
- RFRs are prone to upward/downward spikes at regulatory reporting dates: SOFR is on average 20.25 bps higher at quarter-ends compared to other dates (source: Klingler and Syrstad (2021), period: 08/2014 - 12/2019).

These facts bring evidence of the presence of **stochastic discontinuities**:
new information arriving at pre-determined dates that affects the level of the rates.

SOFR behavior



SOFR time series from 01/12/2015 until 30/09/2019 (source: FED).

SOFR behavior

- Let us consider the spike observed on September 17, 2019.
  According to Anbil et al. (2020):

  > *Strains in money markets in September seem to have originated from routine market events, including a corporate tax payment date and Treasury coupon settlement. The outsized and unexpected moves in money market rates were amplified by a number of factors.*

- This analysis of Anbil et al. (2020) suggests that the date of the spike was known in advance, while the size of the jump was obviously not predictable.

A short overview of the literature on RFR modelling

- General aspects of the Libor reform: Henrard (2019), Piterbarg (2020), Klingler and Syrstad (2021).
- Mercurio (2018): short rate model for SOFR, adding a deterministic spread to the OIS rate.
- Lyashenko and Mercurio (2019): one of the first and most influential contributions, extending the classical Libor market model.
- Macrina and Skovmand (2020): rational model driven by an affine process.
- Willems (2020): extended SABR model applied to caplet pricing.
- Extensions of the Hull-White model: Hofman (2020), Turfus (2020), Hasegawa (2021), Xu (2022).
- Fontana (2022): general affine models for RFRs and pricing formulae.
- Skov and Skovmand (2021), Skov and Skovmand (2022): multi-factor Gaussian model for SOFR futures.
- Rutkowski and Bickersteth (2021): Vasiček model for SOFR, discussing pricing and hedging in the presence of funding costs and collateralization.

Models with stochastic discontinuities

The papers mentioned in the previous slide do not consider the possible presence of stochastic discontinuities in the RFR dynamics. This phenomenon is however playing an important role in recent works:

- Andersen and Bang (2020): spikes in the SOFR dynamics, both at totally inaccessible times and at anticipated times.
- Gellert and Schlögl (2021): a diffusive HJM model for instantaneous forward rates is compatible with the presence of jumps/spikes at fixed times in the short rate, consistently with the empirical evidence on SOFR.
- Backwell and Hayes (2021): a short-rate model for the SONIA rate, based on a pure jump process with predetermined jumps times.

The overnight rate numéraire

Let $(\Omega, \mathcal{F}, \mathbb{F}, Q)$ be a stochastic basis supporting all processes introduced below.

- Let us consider an overnight rate $(\rho_{t_i})_{i=1,\dots,N}$;
- It seems natural to assume that the numéraire S^0 is generated by a roll-over strategy investing in the overnight rate, meaning that

$$S^0_t = \prod_{t_i \leq t} e^{\rho_{t_i}(t \wedge t_{i+1} - t_i)} = \exp\left(\int_0^{\cdot} \rho_t \, dt \right).$$

- More generally, we consider a continuous-time RFR process $\rho = (\rho_t)_{t \geq 0}$ and assume that

$$S^0_t = \exp\left(\int_0^{\cdot} \rho_t \, \eta(dt) \right),$$

  where $\eta(dt) = dt + \sum_{i=1}^N \delta_{t_i}(dt)$.
- The set $\mathcal{T} = \{t_1, \dots, t_N\}$ represent the set of dates at which jumps in the numéraire are expected, due to the roll-over mechanism or for other causes.

Notions of interest rates

Consider a family of tenors $\mathcal{D} = \{\delta_1, \ldots, \delta_m\}$, with $0 < \delta_1 < \ldots < \delta_m$, for $m \in \mathbb{N}$.

- The compounded *setting-in-arrears rate* $R(T, T+\delta)$ is

$$R(T, T+\delta) := \frac{1}{\delta}\left(\frac{S^0_{T+\delta}}{S^0_T} - 1\right) = \frac{1}{\delta}\left(e^{\int_{(T,T+\delta]} \rho_t \eta(dt)} - 1\right).$$

- According to the ISDA protocol, the rate $R(T, T+\delta)$ represents the LIBOR fallback, up to an additive spread determined on the basis of historical data.
- This rate is backward-looking, since its value is only known at $T + \delta$.
- The *forward-looking rate* $F(T, T+\delta)$ is the rate K such that the swaplet delivering $\delta(R(T, T+\delta) - K)$ at $T + \delta$ has zero value at T.
  By definition, the forward-looking rate is $\mathcal{F}_T$-measurable.

Notions of interest rates: forward rates

In this setting, one can consider two different types of forward rates:

1. *backward-looking forward rate* $R(t, T, \delta)$: the rate K such that the swaplet delivering $\delta(R(T, T+\delta) - K)$ at $T+\delta$ has zero value at t;
2. *forward-looking forward rate* $F(t, T, \delta)$: the rate K such that the swaplet delivering $\delta(F(T, T+\delta) - K)$ at $T+\delta$ has zero value at t.

We immediately see that

$$F(t, T, \delta) = R(t, T, \delta), \qquad \text{for all } t \in [0, T].$$

However, while the forward-looking forward rate $F(t, T, \delta)$ stops evolving at T, the backward-looking forward rate $R(t, T, \delta)$ continues to evolve until $T+\delta$, reaching the terminal condition

$$R(T+\delta, T, \delta) = R(T, T+\delta).$$

As in Lyashenko and Mercurio (2019), forward-looking and backward-looking forward rates can be consolidated into a single process $R(\cdot, T, \delta)$. We call this process the *forward term rate*.

The ingredients of a term structure model

Key ingredients of a general term structure model with overnight rates:

1. ZCB prices $P(\cdot, T)$, for all $T \in \mathbb{R}_+$;
2. forward term rates $R(\cdot, T, \delta)$, for all $(T, \delta) \in \mathbb{R}_+ \times \mathcal{D}$.

Remarks:

- ZCBs can be statically replicated from swaplets. This fact implies a model-free representation of swaplet prices in terms of $P(\cdot, T)$ and $R(\cdot, T, \delta)$.
- The interpretation of the forward term rates is flexible:
  - $R(\cdot, T, \delta)$ can represent precisely the backward-looking forward rate...
  - ...but it can also represent a generic LIBOR fallback.
- The last fact is important, since the adoption of a suitable forward-looking credit-sensitive term rate is currently one of the biggest challenges in the market and several candidates exist (e.g., AMERIBOR, AXI, BSBY).
- This suggests that post-reform markets will likely exhibit multi-curve features.

An extended HJM framework - ZCB prices

We start by specifying ZCB prices as follows:

$$P(t,T) = \exp\Big(-\int_{(t,T]} f(t,u)\eta(du)\Big), \qquad \text{for all } 0 \le t \le T,$$

where we recall that $\eta(dt) = dt + \sum_{i=1}^N \delta_{t_i}(dt)$, and we assume that

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \varphi(s,T)dW_s + V(t,T), \qquad \text{for all } 0 \le t \le T,$$

where W is a d-dimensional Brownian motion and $V(\cdot,T) = (V(t,T))_{t\in[0,T]}$ is an adapted pure jump process such that

$$\{\Delta V(\cdot,T) \neq 0\} \subseteq \Omega \times \mathcal{S}, \qquad \text{where } \mathcal{S} = \{s_1,\dots,s_M\}.$$

The set $\mathcal{S}$ includes the dates at which the RFR and forward term rates are expected to jump (can be generalized to predictable times).

Remarks:

- We do not exclude the case $\mathcal{S} \cap \mathcal{T} \neq \emptyset$, where we recall $\mathcal{T} = \{t_1,\dots,t_N\}$.
- The setting can be extended to include totally inaccessible jump terms.

Technical assumptions - ZCB prices

The following conditions hold a.s.:

(i) the *initial forward curve* $T \to f(0,T)$ is $(\mathcal{F}_0 \otimes \mathcal{B}(\mathbb{R}_+))$-measurable, real-valued and satisfies $\int_0^T |f(0,u)|du < +\infty$, for all $T > 0$;

(ii) the *drift process* $\alpha : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is progressively measurable, satisfies $\alpha(t,T) = 0$ for $T < t$, and

$$\int_0^T \int_0^u |\alpha(s,u)| ds \, \eta(du) < +\infty, \qquad \text{for all } T > 0;$$

(iii) the *volatility process* $\varphi : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}^d$ is progressively measurable and satisfies $\varphi(t,T) = 0$ for $T < t$, and

$$\sum_{i=1}^d \int_0^T \left(\int_0^u |\varphi^i(s,u)|^2 ds \right)^{1/2} \eta(du) < +\infty, \qquad \text{for all } T > 0;$$

(iv) the *discontinuity process* $V(\cdot, T) = (V(t,T))_{t \in [0,T]}$ satisfies $\int_0^T |\Delta V(s,u)| du < +\infty$ for all $s \in \mathcal{S}$ and $\Delta V(t,T) = 0$ for $T < t$.

An extended HJM framework - forward term rates

For all $T \in \mathbb{R}_+$ and $\delta \in \mathcal{D}$, we complete the description of the model by

$$R(t,T,\delta) = R(0,T,\delta) + \int_0^t \alpha^R(s,T,\delta)ds + \int_0^t \varphi^R(s,T,\delta)dW_s + V^R(t,T,\delta),$$

for all $0 \leq t \leq T$, where $\alpha^R(\cdot,T,\delta)$ and $\varphi^R(\cdot,T,\delta)$ satisfy standard integrability conditions and $V^R(\cdot,T,\delta)$ is a pure jump process such that

$$\{\Delta V^R(\cdot,T,\delta) \neq 0\} \subseteq \Omega \times (\mathcal{S} \cap [0,T]).$$

An extended HJM framework

Objective: characterize when Q is a risk-neutral measure. This suffices to ensure absence of arbitrage in the sense of NAFLVR for the term structure model, with respect to the numéraire S^0 generated by the overnight rate ρ.

For all $0 \leq t \leq T < +\infty$, we define

$$\bar{\alpha}(t, T) := \int_{[t,T]} \alpha(t, u)\eta(du),$$

$$\bar{\varphi}(t, T) := \int_{[t,T]} \varphi(t, u)\eta(du),$$

$$\bar{V}(t, T) := \int_{[t,T]} \Delta V(t, u)\eta(du).$$

HJM conditions - Part I (ZCB prices)

Theorem

For every $T \geq 0$, $P(\cdot, T)/S^0$ is a Q-local martingale if and only if (integrability condition holds) and the following four conditions hold:

(i) $f(t,t) = \rho_t$, $dt \otimes dQ$-a.e.,

(ii) on $[0, T]$, it holds $dt \otimes dQ$-a.e. that

$$\bar{\alpha}(t, T) = \frac{1}{2}\|\bar{\varphi}(t, T)\|^2$$

(iii) for every $j = 1, \dots, N$ it holds a.s. that

$$f(t_j-, t_j) = \rho_{t_j-} - \log\big(E[e^{-\Delta\rho_{t_j}} | \mathcal{F}_{t_j-}]\big),$$

(iv) for every $i = 1, \dots, M$ it holds a.s. that

$$E\Big[e^{-\Delta\rho_{s_i}\delta_{\mathcal{T}}(s_i)}\big(e^{-\int_{(s_i,T]}\Delta V(s_i,u)\eta(du)} - 1\big)\Big|\mathcal{F}_{s_i-}\Big] = 0.$$

Remark: if $\mathcal{S} \cap \mathcal{T} = \emptyset$, then conditions (i) and (iii) can be jointly written as

$$f(t,t) = \rho_t, \;\; \eta(dt) \otimes dQ\text{-a.e.}$$

HJM conditions - Part II (forward term rates)

Theorem

Q is a risk-neutral measure if and only if all conditions of the previous theorem hold true and, in addition, the following two conditions hold:

(i) on $[0, T]$ it holds $dt \otimes dQ$-a.s. that

$$\alpha^R(t, T, \delta) = \bar{\varphi}(t, T + \delta)^\top \varphi^R(t, T, \delta)$$

(ii) for every $i = 1, \ldots, M$ it holds a.s. that

$$E\Big[\Delta V^R(s_i, T, \delta) e^{-\int_{(s_i, T+\delta]} \Delta V(s_i, u)\eta(du) - \Delta\rho_{s_i}\delta_{\mathcal{T}}(s_i)} \Big| \mathcal{F}_{s_i-}\Big] = 0.$$

A compatibility issue

- So far, we have postulated generic dynamics for a forward term rate $R(\cdot, T, \delta)$ and derived the corresponding no-arbitrage restrictions.
- This is what one can do for forward-looking forward rates as well as for generic credit-sensitive term rates replacing LIBOR.
- However, if $R(\cdot, T, \delta)$ is meant to be the backward-looking forward rate modelled on the whole interval $[0, T+\delta]$, then it must satisfy

$$R(T+\delta, T, \delta) = R(T, T+\delta) = \frac{1}{\delta}\left(\frac{S^0_{T+\delta}}{S^0_T} - 1\right).$$

- This generates a compatibility issue between the following:
  1. the dynamics of $R(\cdot, T, \delta)$ must be consistent with no-arbitrage;
  2. the terminal condition $R(T+\delta, T, \delta) = R(T, T+\delta)$ must be satisfied.

A martingale representation assumption

For simplicity, let us introduce the following assumption.

Assumption

There exists a family $(\xi_1, \dots, \xi_M)$ of random variables on $(\Omega, \mathcal{F}, Q)$ taking values in some space $(B, \mathcal{B}(B))$ such that ξ_i is $\mathcal{F}_{s_i}$-measurable, for each $i = 1, \dots, M$, and every local martingale $N = (N_t)_{t \geq 0}$ on $(\Omega, \mathbb{F}, Q)$ can be represented as

$$N = N_0 + \int_0^{\cdot} \theta_t dW_t + \sum_{i=1}^{M} f_i(\xi_i) \mathbf{1}_{[\![s_i, +\infty[\![},$$

where, for all $i = 1, \dots, M$, $f_i(\cdot) : \Omega \times B \to \mathbb{R}$ is a $(\mathcal{F}_{s_i-} \otimes \mathcal{B}(B))$-measurable function such that

$$E[f_i(\xi_i) | \mathcal{F}_{s_i-}] = 0 \quad \text{a.s.}$$

We denote by $\mathcal{H}$ the space of all such functions $f = (f_1, \dots, f_M)$.

A compatibility issue - a BSDE viewpoint

Under the assumption that the family of ZCB prices is arbitrage-free, the following result gives a BSDE characterization of compatible forward term rate processes.

Theorem

A forward term rate process $R(\cdot, T, \delta)$ is compatible if and only if $R(\cdot, T, \delta) = Y$, where $(Y, z, w) \in \mathcal{S}_p([0, T+\delta]) \times L^2_{\mathrm{loc}}([0, T+\delta]) \times \mathcal{H}$ is a solution to the BSDE

$$\begin{cases} dY_t = \bar{\varphi}(t, T+\delta)^\top z_t dt - \sum_{i=1}^{M} E[e^{-\bar{V}(s_i, T+\delta)} w_i(\xi_i) | \mathcal{F}_{s_i-}] \delta_{s_i}(dt) \\ \qquad + z_t dW_t + \sum_{i=1}^{M} w_i(\xi_i) \delta_{s_i}(dt), \\ Y_{T+\delta} = R(T, T+\delta). \end{cases}$$

A compatibility issue - a BSDE viewpoint

In general, the above BSDE may admit multiple solutions:

- the usual definition of simply compounded forward-rates

$$R(t,T,\delta) := \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right)$$

  generates a compatible forward term rate process;

- if $1/S^0_T \in L^1(Q)$, for all $T > 0$, then the process

$$R'(t,T,\delta) := \frac{1}{\delta}\frac{S^0_t}{P(t,T+\delta)}E\left[\frac{1}{S^0_T} - \frac{1}{S^0_{T+\delta}}\middle|\mathcal{F}_t\right]$$

  generates a compatible forward term rate process.

- R and R' coincide whenever ZCB prices are *true* martingales under Q, but nevertheless there may exist other compatible forward term rate processes.
- Moreover, one can prove that, under suitable integrability assumptions, R' is the unique BSDE solution in the space of $Q^{T+\delta}$-martingales .

Risk-neutral pricing

The last observation enables us to recover the usual risk-neutral representation of backward-looking forward rates, first obtained in Lyashenko and Mercurio (2019).

Corollary

Suppose that S^0-discounted ZCB and swaplet prices are true martingales under Q. Then, for all $T \geq 0$ and $\delta \in \mathcal{D}$, it holds that

$$P(t,T) = E\Big[e^{-\int_{(t,T]} \rho_s \eta(ds)}\Big|\mathcal{F}_t\Big], \qquad \text{for } 0 \leq t \leq T,$$

$$R(t,T,\delta) = \begin{cases} \frac{1}{\delta}\left(\frac{P(t,T)}{P(t,T+\delta)} - 1\right), & \text{for } 0 \leq t \leq T, \\ \frac{1}{\delta}\left(\frac{e^{\int_{(T,t]} \rho_s \eta(ds)}}{P(t,T+\delta)} - 1\right), & \text{for } T < t \leq T+\delta. \end{cases}$$

The affine framework

The presence of stochastic discontinuities requires an extension of affine processes: affine semimartingales generalize affine processes by allowing for jumps at fixed points in time with possibly state-dependent jump sizes.

⇒Keller-Ressel et al. (2019).

An affine semimartingale $X = (X_t)_{t\geq 0}$ taking values in $D := \mathbb{R}^m_+ \times \mathbb{R}^n$ satisfies

$$E\big[e^{\langle u, X_T\rangle}|\mathcal{F}_t\big] = \exp\big(\phi_t(T,u) + \langle \psi_t(T,u), X_t\rangle\big), \tag{1}$$

$0 \leq t \leq T < +\infty$ and $u \in \mathcal{U} = \mathbb{C}^m_- \times \mathrm{i}\mathbb{R}^n$, where the functions $\phi_t(T,u)$ and $\psi_t(T,u)$ take values in $\mathbb{C}$ and $\mathbb{C}^d$ and satisfy certain generalized Riccati equations.

The affine framework

Definition

An term structure model is called *affine* if

$$f(t,T) = f(0,T) + \int_0^t \zeta(s,T)dX_s,$$

$$R(t,T,\delta) = R(0,T,\delta) + \int_0^t \zeta^R(s,T,\delta)dX_s,$$

for all $0 \leq t \leq T < +\infty$ and $\delta \in \mathcal{D}$, where $\zeta : \Omega \times \mathbb{R}^2_+ \to \mathbb{R}^d$ and $\zeta^R : \Omega \times \mathbb{R}^2_+ \times \mathcal{D} \to \mathbb{R}^d$ are predictable processes such that $\zeta(\cdot,T), \zeta^R(\cdot,T,\delta) \in L(X)$ and the stochastic integrals $\zeta(\cdot,T)\cdot X$ and $\zeta^R(\cdot,T,\delta)\cdot X$ are special semimartingales.

By relying on the previous results, one can obtain a characterization of affine term structure models that are free of arbitrage.

Short-rate approach: let the RFR be given by

$$\rho_t = \ell(t) + \langle \Lambda, X_t \rangle, \qquad \text{for all } t \geq 0.$$

Adopting this short-rate formulation, we can prove that

- the process $(X, \int_0^{\cdot} \rho_t \eta(dt))$ is an affine semimartingale, its characteristic function can be computed in terms of the semimartingale characteristics of X.

An example: an extended Hull-White model

Assume that $\rho = (\rho_t)_{t \geq 0}$ satisfies

$$d\rho_t = \big(\alpha(t) + \beta\rho_t\big)dt + \sigma dW_t + dJ_t,$$

where J is a pure jump process independent of W:

$$J = \sum_{i=1}^{M} \xi^i \mathbf{1}_{[\![s_i,+\infty[\![},$$

Integrating, it holds that

$$\rho_T = \rho_t e^{\beta(T-t)} + \int_t^T e^{\beta(T-s)}\alpha(s)ds + \sigma e^{\beta T}\int_t^T e^{-\beta s}dW_s$$
$$+ \sum_{i=1}^{M} \mathbf{1}_{s_i \in (t,T]}\xi^i e^{\beta(T-s_i)}.$$

and the Riccati equations can be explicitly solved.

In the Gaussian case (i.e., $(\xi_i)_{i=1,\dots,M}$ independent and Gaussian):

- explicit formula for ZCB prices;
- Black-Scholes-type formula for RFR caplet and similar payoffs.

An example: an extended Hull-White model

Considering a two-dimensional extended Hull-White model, we can generate different types of jumps (spikes and structural jumps).

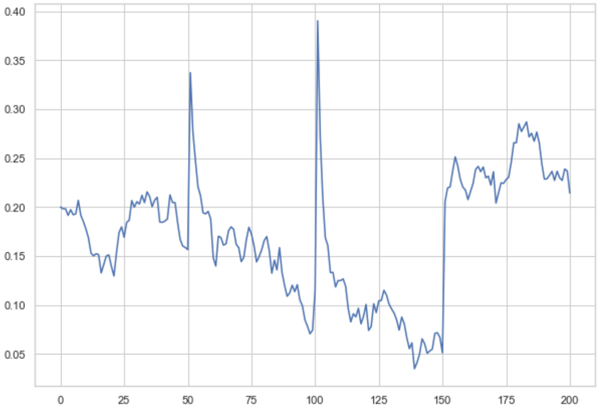


Figure: The simulated path exhibits spikes at the discontinuity dates $t = 50$ and $t = 100$ and a structural jump at $t = 150$.

Hedging with stochastic discontinuities

- Stochastic discontinuities induce market incompleteness.
- We shall therefore rely on the concept of local risk-minimization.

For simplicity, we work under the same martingale representation assumption as before: every local martingale N can be represented as

$$N = N_0 + \int_0^{\cdot} \theta_t dW_t + \sum_{i=1}^{M} f_i(\xi_i) \mathbf{1}_{[\![s_i, +\infty [\![}.$$

- Again for simplicity, suppose that the market contains a single traded asset with discounted price process X, assumed to be a special semimartingale:

$$X = X_0 + A + M,$$

  where
  - A is predictable process of finite-variation,
  - $M = \int_0^{\cdot} \eta_t dW_t + \sum_{s_i \leq \cdot} w_i(\xi_i)$ is a square-integrable martingale,
  - $A_0 = M_0 = 0$.
- For instance, X can represent the price process of a SOFR future contract (currently the most liquid SOFR product).

Hedging with stochastic discontinuities

Let $H \in L^2(Q)$ be an $\mathcal{F}_T$-measurable payoff. We denote by Θ the set of predictable processes ζ such that $E[\int_0^T \zeta_u^2 d\langle M\rangle_u + (\int_0^T \zeta_u dA_u)^2] < +\infty$.

Definition

- We call *H-admissible strategy* a pair $\varphi = (\zeta, V)$, where $\zeta = (\zeta_t)_{t\in[0,T]} \in \Theta$ and $V = (V_t)_{t\in[0,T]}$ is an adapted process such that $V_T = H$ a.s.
- We say that an H-admissible strategy $\varphi = (\zeta, V)$ is *locally risk-minimizing* if the associated cost process

$$C_t(\varphi) := V_t - \int_0^t \zeta_u dX_u, \qquad \text{for all } t \in [0,T],$$

is a square-integrable martingale strongly orthogonal to M.

Remarks:

- ζ_t and V_t represent respectively the positions held in the traded security and the portfolio value at time t, for all $t \in [0,T]$.
- in the present context, it can be shown that the above definition is equivalent to the original definition of Schweizer (1991).

Hedging with stochastic discontinuities

- By absence of arbitrage, there exists a predictable process λ such that $A = \int_0^{\cdot} \lambda_u d\langle M\rangle_u$. In particular, this implies that

$$\Delta A_{s_i} = \lambda_{s_i} E[(\Delta M_{s_i})^2 | \mathcal{F}_{s_i-}] \text{ a.s., for all } i = 1, \dots, M.$$

- Assume that $\widehat{Z} := \mathcal{E}(-\int_0^{\cdot} \lambda_u dM_u)$ is a strictly positive square-integrable martingale and define the minimal martingale measure by $d\widehat{Q} = \widehat{Z}_T dQ$.
- We can then define the $\widehat{Q}$-martingale $\widehat{H} = (\widehat{H}_t)_{t\in[0,T]}$ by

$$\widehat{H}_t := \widehat{E}[H|\mathcal{F}_t], \qquad \text{for all } t \in [0,T],$$

where we denote by $\widehat{E}$ the expectation with respect to $\widehat{Q}$.
- By Bayes' formula, $\widehat{H} = N/\widehat{Z}$, with $N_t := E[\widehat{Z}_T H|\mathcal{F}_t]$, for all $t \in [0,T]$.
- As a consequence of the martingale representation assumption, we have that

$$N = N_0 + \int_0^{\cdot} \theta_u dW_u + \sum_{s_i \le \cdot} \Delta N_{s_i}.$$

Hedging with stochastic discontinuities

Proposition

Let H be an $\mathcal{F}_T$-measurable random variable and $\sup_{t\in[0,T]} \widehat{H}_t \in L^2(P)$. Define the predictable process

$$\zeta_t^H := \left(\widehat{Z}_{t-}^{-1}\eta_t^{-1}\theta_t + \widehat{H}_{t-}\lambda_t\right)\delta_{\mathcal{S}^c}(t) + \frac{E[\Delta\widehat{H}_t\Delta M_t|\mathcal{F}_{t-}]}{E[(\Delta M_t)^2|\mathcal{F}_{t-}]}\delta_{\mathcal{S}}(t).$$

If $\zeta^H \in \Theta$, then an H-admissible locally risk-minimizing strategy is given by $\varphi^H = (\zeta^H, V^H)$, where $V_t^H = \widehat{H}_t$, for all $t \in [0,T]$.

Remarks:

- perfect replication at all times $t \in [0,T] \setminus \mathcal{S}$, when the only active source of randomness is the Brownian motion W;
- at the discontinuity dates $\mathcal{S} = \{s_1, \ldots, s_M\}$, the strategy $\zeta_{s_i}^H$ is determined by a linear regression of $\Delta\widehat{H}_{s_i}$ onto ΔX_{s_i}, conditionally on $\mathcal{F}_{s_i-}$:

$$\zeta_{s_i}^H = \frac{\mathrm{Cov}(\Delta\widehat{H}_{s_i}, \Delta X_{s_i}|\mathcal{F}_{s_i-})}{\mathrm{Var}(\Delta X_{s_i}|\mathcal{F}_{s_i-})},$$

- In the paper, explicit formula for the locally risk-minimizing strategy of a SOFR caplet with respect to a SOFR future.

Conclusions

- Stochastic discontinuities as an essential feature of interest rate markets;
- consistent modelling of overnight RFRs together with forward term rates;
- affine semimartingales as a tractable class of driving processes;
- what kind of forward term rate will reach market consensus?
- new types of interest rate derivatives?

Thank you for your attention!

For more information:

C. Fontana, Z. Grbac, T. Schmidt (2022), Term structure modelling with overnight rates beyond stochastic continuity, available on arXiv and SSRN.

Anbil, S., Anderson, A. and Senyuz, Z. (2020), 'What happened in money markets in september 2019?', https://www.federalreserve.gov/econres/notes/feds-notes/what-happened-in-money-markets-in-september-2019-20200227.htm.

Andersen, L. and Bang, D. (2020), 'Spike modeling for interest rate derivatives with an application to SOFR caplets', Preprint (available at https://ssrn.com/abstract=3700446.

Backwell, A. and Hayes, J. (2021), 'Expected and unexpected jumps in the overnight rate: consistent management of the Libor transition', Preprint (available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3989108).

Fontana, C. (2022), 'Caplet pricing in affine models for risk-free rates', In preparation.

Gellert, K. and Schlögl, E. (2021), 'Short rate dynamics: A fed funds and SOFR perspective', Preprint (available at https://arxiv.org/abs/2101.04308.

Hasegawa, T. (2021), 'Caplet formulae for backward-looking term rates with Hull-White model', Working paper (available at https://papers.ssrn.com/abstract_id=3909949.

Henrard, M. (2019), 'LIBOR fallback and quantitative finance', *Risks* **7**(3), 88.

Hofman, K. (2020), 'Implied volatilities for options on backward-looking term rates', Preprint (available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3593284.

Keller-Ressel, M., Schmidt, T. and Wardenga, R. (2019), 'Affine processes beyond stochastic continuity', *Annals of Applied Probability* **29**(6), 3387–3437.

Klingler, S. and Syrstad, O. (2021), 'Life after LIBOR', *Journal of Financial Economics* **141**(2), 783–801.

Lyashenko, A. and Mercurio, F. (2019), Looking forward to backward-looking rates: a modeling framework for term rates replacing LIBOR. Working paper (available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3330240).

Macrina, A. and Skovmand, D. (2020), 'Rational savings account models for backward-looking interest rate benchmarks', *Risks* **8**(1), 23.

Mercurio, F. (2018), 'A simple multi-curve model for pricing SOFR futures and other derivatives', Preprint (available at https://ssrn.com/abstract=3225872).

Piterbarg, V. (2020), 'Interest rates benchmark reform and options markets', Preprint (available at https://ssrn.com/abstract=3537925.

Rutkowski, M. and Bickersteth, M. (2021), 'Pricing and hedging of SOFR derivatives under differential funding costs and collateralization'. Preprint (available at https://arxiv.org/abs/2112.14033).

Schweizer, M. (1991), 'Option hedging for semimartingales', *Stochastic Processes and their Applications* **37**, 339–363.

Skov, J. and Skovmand, D. (2021), 'Dynamic term structure models for SOFR futures', *Journal of Futures Markets* **41**(10), 1520–1544.

Skov, J. and Skovmand, D. (2021), 'Decomposing LIBOR in transition: evidence from the futures markets', Preprint (available at https://papers.ssrn.com/abstract=4011937)

Turfus, C. (2020), 'Caplet pricing with backward-looking rates', Preprint. Available at SSRN: https://ssrn.com/abstract=3527091.

Willems, S. (2020), 'SABR smiles for RFR caplets', Working paper (available at https://ssrn.com/abstract=3567655.

Xu, M. (2022), 'SOFR derivative pricing using a short rate model', Working paper (available at https://papers.ssrn.com/abstract_id=4007604.