Moral hazard for time-inconsistent agents, BSVIEs and stochastic targets

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Outline

1. The big picture
   - Motivating example
   - Time-inconsistency
2. Main results
   - An extended DPP
   - The characterising BSDE system
3. Back to contract theory

The setting

- Setting similar to Hölmstrom and Milgrom (1987). Agent chooses control $\alpha \in \mathcal{A}$, and induces probability measure $\mathbb{P}^\alpha$ such that

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• Principal rewards Agent at time T with $\xi(X_{\cdot \wedge T})$. Agent's criterion is

$$J(t, \alpha, \xi) := \mathbb{E}^{\mathbb{P}^\alpha}\left[f(T-t)\xi(X_{\cdot \wedge T}) - \int_t^T f(s-t)\frac{\alpha_s^2}{2}\mathrm{d}s \middle| \mathcal{F}_t \right],$$

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with $f(0) = 1$.

• Given effort α and contract ξ, Principal's criterion is

$$\mathbb{E}^{\mathbb{P}^\alpha}\left[X_T - \xi(X_{\cdot\wedge T}) \right].$$

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- Stochastic representation for the value function of Agent, and thus for $\xi(X_{\cdot\wedge T})$.

- Principal then maximises his criterion over Z, optimal contract is

$$\xi(X_{\cdot\wedge T}) = C + X_T,$$

with C chosen to ensure that Agent accepts the contract.

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  (ii) Second task: understand problem faced by Principal $\Longrightarrow$ doable in the example, general case still open, but seems linked to optimal control of stochastic Volterra equations.

The basic problem

- On space $(\Omega, \mathcal{F})$, let $\mathbb{P}^\nu$ be a weak solution to the controlled SDE

$$X_t = x + \int_0^t b_r(X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + \int_0^t \sigma_r(X_{r\wedge\cdot})\mathrm{d}W_r,\ t \in [0, T].$$

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- The reward functional

$$v(t,\nu) := J(t,t,\nu) = \mathbb{E}^{\mathbb{P}^\nu}\left[\int_t^T f_r(t, X_{r\wedge\cdot}, \nu_r)\mathrm{d}r + F(t, X_{T\wedge\cdot})\middle|\mathcal{F}_t\right].$$

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- Control problem: Since DPP fails, look for equilibria.

Outline

An extended DPP

Though classical DPP does not hold, can obtain

Theorem [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium then

$$v(0,\nu^\star) = \sup_{\nu \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^\nu}\bigg[v(\tau,\nu^\tau) + \int_0^\tau \bigg(f_r(r,X_{r\wedge\cdot},\nu_r) - \frac{\partial F}{\partial s}(r) - \int_r^T \frac{\partial f_r}{\partial s}(u,X_{r\wedge\cdot},\nu_u^\star)\mathrm{d}u\bigg)\mathrm{d}r\bigg].$$

HJB BSDE system

Define the Hamiltonian

$$H_t(x,z) := \sup_{a\in A}\{h_t(t,x,z)\},\ h_t(s,x,z,a) := f_t(s,x,a) + b_t(x,a)\cdot z,$$

$$\nu_t^\star(x,z) := \mathrm{argmax}_{a\in A}\{h_t(t,x,z,a)\}.$$

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Extended DPP relates value of the agent at equilibrium to

$$\begin{cases} Y_t = F(T, X_{\cdot\wedge T}) + \int_t^T \big(H_r(X,Z_r) - \partial Y_r^r\big)\mathrm{d}r - \int_t^T Z_r\cdot \mathrm{d}X_r, \\ Y_t^s = F(s, X_{\cdot\wedge T}) + \int_t^T h_r(s, X, \nu_r^\star(X_{r\wedge\cdot}, Z_r), Z_r^s)\mathrm{d}r - \int_t^T Z_r^s\cdot \mathrm{d}X_r, \end{cases}$$

where

$$\partial Y_t^s := \frac{\partial}{\partial_s} Y_t^s.$$

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For exponential discounting, $\partial Y_t^s = \delta Y_s \Longrightarrow$ no coupling and classical HJB BSDE.

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- Earlier contributions argued by passing informally to the limit from discrete-time. Thanks to extended DPP

Theorem [Hernández, P. (2019)]

If $\nu^\star$ is an equilibrium, then there exists a solution to the BSDE system, and necessarily $\nu^\star = \nu^\star(X, Z)$.

Outline

Agent's problem

- Agent looks for equilibria, so that Principal can only offer $\xi(X_{\cdot\wedge T})$ such that at least one exists. By our results, equilibria exist if and only if we have a solution to

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the equilibrium is $\nu^\star_\cdot := Z_\cdot$, and $J(t,\alpha^\star,\xi) = Y_t = Y_t^t$.

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- Infinitely many representations for $\xi(X_{\cdot\wedge T})$...

- ...cannot use only one and optimise over Z as before, need to understand relationships between Z and Z^s.

The BSDE system

- We can directly check that

$$Z_t = Z_t^t, \text{ and } Z_t^s = \frac{f(T-s)}{f(T)} Z_t^0 + f(T-s)\widetilde{Z}_t^s,$$

where $\widetilde{Z}^s$ appears in the following martingale representation

$$M_t^s := \mathbb{E}^{\mathbb{P}^{\nu^\star}}\left[\int_0^T \left(\frac{f(r)}{f(T)} - \frac{f(r-s)}{f(T-s)}\right)\frac{Z_r^2}{2}\mathrm{d}r \middle| \mathcal{F}_t\right] = M_0^s + \int_0^t \widetilde{Z}_r^s \mathrm{d}X_r.$$

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• It also vanishes whenever Z is deterministic!

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$$\xi^\star := C + \int_0^T \frac{f(T)}{f(r)f(T-r)} \mathrm{d}X_r.$$

- Can check directly that this is an admissible contract which attains an upper bound for Principal's value.

The general case?

In great generality, system is equivalent to solving, $(s,t)\in[0,T]^2$

$$Y_t^s = \mathrm{U_A}(s,\xi) + \int_t^T h_r^\star(s, X_{r\wedge\cdot}, Y_r^s, Z_r^s, Y_r^r, Z_r^r)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r.$$

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$\mathcal{H}^{2,2}$: $Z \in \overline{\mathbb{H}}^{2,2}$ satisfying

$$Y_t^{s,Z} = Y_0^s - \int_0^t h_r^\star(s, X_{\cdot\wedge r}, Y_r^{s,Z}, Z_r^s, Y_r^{r,Z}, Z_r^r)\mathrm{d}r + \int_0^t Z_r^s \mathrm{d}X_r,\ (s,t)\in[0,T]^2.$$
$$\mathrm{U_A}^{(-1)}(s, Y_T^s) = \mathrm{U_A}^{(-1)}(u, Y_T^u),\ (s,u)\in[0,T]^2. \tag{3}$$

To sum up

- Insight: V^{A} solves a backward stochastic Volterra integral equation.
- Principal's problem boils to optimal control of a Volterra forward equation (1) with Volterra controls (2), and stochastic target constraints (3) i.e. a family $(Z_t^s)_{(s,t)\in[0,T]^2}$, (3) holds

$$Y_t^s = \mathrm{U_A}(s,\xi) + \int_t^T h_r^\star\big(s, X_{r\wedge\cdot}, Y_r^s, Z_r^s, Y_r^r, Z_r^r\big)\mathrm{d}r - \int_t^T Z_r^s \mathrm{d}X_r.$$

- (1) is known, (2) is expected to be a slight generalisation, (3) is the really hard part: need to understand Volterra stochastic target problems, and optimal control of Volterra processes with state constraints .

A little extra

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- constraints are put on payments ξ;
- Agent has mean–variance type criteria;
- more surprisingly: when Agent and Principal play a Nash game between themselves;
- conjecture this holds true generically for Stackelberg games in continuous-time: problem of the leader boils down to optimal control with stochastic target constraints.

Thank you for your attention!