Speeding up the Euler scheme for killed diffusions

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Outline of talk

Background on linear diffusions

Setup

- X is a regular diffusion on $(\ell, r) \subset \mathbb{R}$ adapted to right continuous $(\mathcal{F}_t)_{t \geq 0}$. ℓ and r are allowed to be infinite.
- If any of the boundaries are reached in finite time, the process is killed and sent to the cemetery state, Δ, i.e no reflecting boundaries.

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- We shall write $E^x[Y]$ to denote expectation of Y with respect to P^x. Recall that Markov property means $E^x[f(X_{t+s})|\mathcal{F}_t] = E^{X_t}[f(X_s)]$ while the strong Markov property amounts to $E^x[f(X_{T+s})|\mathcal{F}_T] = E^{X_T}[f(X_s)]$ for all stopping times.

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- Diffusion assumption entails X is continuous and strong Markov, while the regularity amounts to $P^x(T_y < \infty) > 0$ whenever x and y belongs to the open interval (l, r), where $T_y := \inf\{t > 0 : X_t = y\}$ for $y \in (l, r)$.

1. Brownian motion living on an interval (a, b) and killed as soon as it reaches the boundary.
2. δ-dimensional Bessel process on $(0, \infty)$.
3. Solution of a stochastic differential equation:

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds, \qquad t < \zeta, \qquad (1)$$

where $\zeta := \inf\{t \geq 0 : X_t \in \{l, r\}\}$.

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More on transience

- A consequence (in fact another formulation) of transience is that for any continuous f compactly supported in (l, r)

$$E^x \int_0^\zeta f(X_t)dt < \infty.$$

- Consequently, there exists a potential kernel, u, such that

$$E^x \int_0^\zeta f(X_t)dt = \int_l^r u(x,y)f(y)m(dy).$$

Moreover, u is continuous and for $x \leq y$,

$$u(x,y) = u(y,x) = \frac{(s(x) - s(\ell))(s(r) - s(y))}{s(r) - s(\ell)}) \leq u(y,y).$$

In particular,

$$P^x(T_y < \infty) = \frac{u(x,y)}{u(y,y)}.$$

Recurrent transformations

Recurrent transformations (Çetin (2018))

Definition 1

Let X be a regular diffusion satisfying (1) and $h : (l, r) \mapsto (0, \infty)$ be a continuous function. (h, M) is said to be a recurrent transform (of X) if the following are satisfied:

1. M is an adapted process of finite variation.
2. $h(X)M$ is a nonnegative local martingale.
3. There exists a unique weak solution to

$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t \left\{ b(X_s) + \sigma^2(X_s)\frac{h'(X_s)}{h(X_s)} \right\} ds. \tag{2}$$

4. The (scale) function s_r is finite for all $x \in (l, r)$ with $-s_r(l+) = s_r(r-) = \infty$, where

$$s_r(x) := \int_c^x \frac{s'(y)}{h^2(y)} dy, \qquad x \in (l, r), \tag{3}$$

"Example" 1

- Suppose X is transient, let $y \in (l, r)$ be fixed and consider

$$h(x) := u(x, y), \; x \in (l, r), \text{ and } M_t = \exp\left(\frac{s'(y)L_t^y}{2u(y, y)}\right).$$

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$$X_t = x + \int_0^t \sigma(X_s)dB_s + \int_0^t \left\{b(X_s) + \sigma^2(X_s)\frac{u_x(X_s, y)}{u(X_s, y)}\right\} ds. \tag{4}$$

- If $R^{h,x}$ denotes the law of the solution and T is an $R^{h,x}$-a.s. finite stopping time, then for any $F \in \mathcal{F}_T$

$$P^x(\zeta > T, F) = u(x, y)E^{h,x}\left[\mathbf{1}_F \frac{1}{u(X_T, y)} \exp\left(-\frac{s'(y)}{2u(y, y)}L_T^y\right)\right]. \tag{5}$$

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An example

- Suppose X is a Brownian motion that is killed when hitting 0 or 1. Then,

$$u(x,y) = x(1-y),\ 0 < x \leq y < 1.$$

- Thus, if we apply the recurrent transform from Example 1 with $y = 1/2$, we obtain the following SDE:

$$dX_t = dB_t + \left\{ \frac{1}{X_t} \mathbf{1}_{[X_t \in (0, \frac{1}{2}]]} - \frac{1}{1-X_t} \mathbf{1}_{[X_t \in (\frac{1}{2}, 1)]} \right\} dt.$$

- Recall that the recurrent transformation implies that the solution to the above SDE never hits 0 or 1, which is also clear from the SDE representation.

Extension to bounded potentials

- Let μ be a Borel probability measure on (l, r) such that $\int_{(l,r)} |s(y)|\mu(dy) < \infty$. Suppose X is transient and define

$$h(x) := \int_{(l,r)} u(x, y)\mu(dy).$$

- (h, M) is a recurrent transform of X, where

$$M_t := \exp\left(\int_0^t \frac{1}{h(X_s)} dA_s\right) \text{ and } A_t := \int_{(l,r)} \frac{s'(x)L_t^x}{2}\mu(dx).$$

- If $R^{h,x}$ denotes the law of the solution of (2) and T is a stopping time such that $R^{h,x}(T < \infty) = 1$, then for any $F \in \mathcal{F}_T$

$$P^x(\zeta > T, F) = h(x)E^{h,x}\left[\mathbf{1}_F \frac{1}{h(X_T)} \exp\left(-\int_0^t \frac{1}{h(X_s)} dA_s\right)\right],$$

where $E^{h,x}$ is the expectation operator with respect to the probability measure $R^{h,x}$.

Explicit Euler-Maruyama schemes for diffusions

The case of no-killing

- Suppose the solution of (1) has infinite lifetime and we are interested in $E^x[g(X_T)]$ for some bounded function g.

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- We in general don't know the transition density explicitly, so we must resort to some approximation algorithms.
- The most popular and straightforward algorithm is the *explicit* Euler-Maruyama scheme:

$$X_{t_n}^N = X_{t_{n-1}}^N + b(X_{t_{n-1}}^N)\frac{T}{N} + \sigma(X_{t_{n-1}}^N)(B_{t_n} - B_{t_{n-1}}), \quad t_n = \frac{nT}{N}, n \in \{0, N\}, \ X_0^N = x. \tag{6}$$

Then, an approximation of $E^x[g(X_T)]$ is found by averaging $g(X_{t_N}^N)$ over a sufficiently large number of simulations.

Convergence rate for the explicit scheme

- A relevant question in above algorithm is ‘how fine do we need to discretize in order to get a ‘negligible’ error for practical purposes?’
- Note that N is the number of discretizations and the ‘weak error’ is given by

$$e(T) = E^x[g(X^N_{t_N})] - E^x[g(X_T)].$$

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- Under some regularity conditions on the coefficients of the SDE and g, there exists a bounded function u with bounded derivatives such that $E^x[g(X_T)|\mathcal{F}_t] = u(t, X_t)$. In particular, $u(T, x) = g(x)$ and

$$u_t + bu_x + \frac{1}{2}\sigma^2 u_{xx} = 0.$$

Convergence rate for the explicit scheme

- Moreover,

$$\begin{aligned} e(T) =& E^x[u(T, X_T^N)] - E^x[u(T, X_T)] \\ =& E^x[u(T, X_T^N)] - u(0, x) \\ =& \sum_{n=0}^{N-1} E^x[u(t_{n+1}, X_{t_{n+1}}^N) - u(t_n, X_{t_n}^N)] \end{aligned}$$

- With the help of Ito's formula, the regularity conditions imply

$$\left| E^x[u(t_{n+1}, X_{t_{n+1}}^N) - u(t_n, X_{t_n}^N)] \right| \leq \frac{K}{N^2},$$

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where K is a constant independent of N.
- Therefore, $|e(T)| \sim O(\frac{1}{N})$.

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- The reason is that (6) produces a process that can exit (ℓ, r), and the most straightforward explicit scheme would be $E^x[g(X_T^N)\mathbf{1}_{[T<\zeta_N]}]$, where ζ_N is the first time that the discretized process exits (ℓ, r).

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- Again one can find a function v vanishing at the accessible boundaries such that such that $e(T) = E^x[u(T, X_T) - u(T, X_T^N)]$. However, u_x does not vanish at the boundaries, and thus, the application of a generalized Ito's formula yields local time terms.
- The local time terms result in a lower weak convergence rate, $O(N^{-1/2})$ (see Gobet (1999)).

A new Euler-Maruyama scheme for killed diffusions

The case of killing

- Now suppose the solution of (1) has a finite lifetime and we are interested in $E^x[g(X_T)\mathbf{1}_{[T<\zeta]}]$ for some bounded function g.
- We can assume without loss of generality that X is on natural scale by considering $s(X)$ if necessary. This amounts to assuming that X is a local martingale, i.e. $b \equiv 0$. Note that there is one-to-one correspondence between X and $s(X)$ since s is strictly increasing.

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- Since at least one of the boundaries is accessible, by considering $-X$ if necessary, we may assume ℓ is an accessible boundary. Moreover, by a further translation, we may assume $\ell = 0$.
- Given the aforementioned problems with killed diffusions, can recurrent transformations help us to improve the convergence rate?

Recurrent transformation via bounded potentials

- Let h be a potential such that $h(x) = \int u(x,y)f(y)m(dy)$, where $f \geq 0$ is continuous and $\int f(y)m(dy)$ as well as $\int f(y)ym(dy)$ are finite. Moreover, $\frac{1}{2}\sigma^2 h'' = -f$.
- h is bounded, concave, and $(h, \exp(\int_0 \frac{f(X_s)}{h(X_s)} ds))$ is a recurrent transformation. The resulting law $R^{h,x}$ is the law of the following process:

$$dX_t = \sigma(X_t)dW_t + \sigma^2(X_t)\frac{h'(X_t)}{h(X_t)}dt. \tag{7}$$

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- Now consider the following explicit scheme

$$X^N_{t_n} = X^N_{t_{n-1}} + \sigma^2(X^N_{t_{n-1}})\frac{h'}{h}(X^N_{t_{n-1}})\frac{T}{N} + \sigma(X^N_{t_{n-1}})(B_{t_n} - B_{t_{n-1}}). \tag{8}$$

Explicit scheme for the recurrent transformation

- If one wants to study the explicit scheme using a PDE method as before, the other object of interest is

$$v(T-t,x) = E^{h,x}\left[\frac{g(X_t)}{h(X_t)}\exp\left(-\int_0^t \frac{\sigma^2(X_s)h''(X_s)}{2h(X_s)}ds\right)\right], x\in(0,r). \tag{9}$$

- Although the numerical experiments converge, there are two immediate difficulties in proving the weak convergence rate for (8):
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- The first issue is somewhat controllable as we shall see later by choosing h accordingly.
- However, the second difficulty does not go away and one needs to impose ad hoc boundary specifications.

A drift-implicit scheme

- As before, let $t_n = \frac{n}{N} T$ for $n = 0, \dots, N$. Set $\widehat{X}_0 = X_0$ and proceed inductively by setting

$$\widehat{X}_t = \widehat{X}_{t_n} + \sigma(\widehat{X}_{t_n})(W_t - W_{t_n}) + (t - t_n)\sigma^2(\widehat{X}_{t_n}) \frac{h'(\widehat{X}_t)}{h(\widehat{X}_t)} \tag{10}$$

for $t \in (t_n, t_{n+1}]$ and $n = 0, \dots N - 1$.

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- We shall call this well-defined scheme *continuous backward Euler-Maruyama (BEM) scheme*.

The first key lemma

Suppose that $h \in C_b^2((0,r),(0,\infty))$, $h^{(3)}$ exists and satisfies $|h^{(3)}| \leq K(1+h^{-p})$ for some constant K and $p \in [0,1)$. Define $H(t_n, z; t, x) = x - \sigma^2(z)(t-t_n)\frac{h'}{h}(x)$. Then for $t \in (t_n, t_{n+1}]$

$$
\begin{aligned}
d\widehat{X}_t &= \frac{\sigma(\widehat{X}_{t_n})}{H_x(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} dW_t \\
&\quad + \frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} \left\{ \frac{h'}{h}(\widehat{X}_t) + \mu(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t) \right\} dt.
\end{aligned}
\tag{11}
$$

The first key lemma

Suppose that $h \in C_b^2((0,r),(0,\infty))$, $h^{(3)}$ exists and satisfies $|h^{(3)}| \leq K(1+h^{-p})$ for some constant K and $p \in [0,1)$. Define $H(t_n, z; t, x) = x - \sigma^2(z)(t-t_n)\frac{h'}{h}(x)$. Then for $t \in (t_n, t_{n+1}]$

$$\begin{aligned} d\widehat{X}_t = & \frac{\sigma(\widehat{X}_{t_n})}{H_x(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} dW_t \\ & + \frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} \left\{ \frac{h'}{h}(\widehat{X}_t) + \mu(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t) \right\} dt. \end{aligned} \tag{11}$$

Consider the sets $O_1 := \{x : h'(x) > 0\}$ and $O_2 := \{x : h'(x) < 0\}$. Then

$$\inf_{x \in O_1} \mu(t_n, z; t, x) \geq c_1 \text{ and } \sup_{x \in O_2} \mu(t_n, z; t, x) \leq c_2$$

for some constants $c_1 \leq 0 \leq c_2$ that do not depend on t_n, t or z.

Speed of weak convergence

- Consider the expected associated error

$$E^{h,X_0}\left[v(T,\widehat{X}_T)\pi_N\right] - v(0,X_0),$$

where

$$\pi_k(s) := \exp\left(\sum_{n=0}^{k-1} s\sigma^2(\widehat{X}_{t_n})\frac{h''(\widehat{X}_{t_n})}{2h(\widehat{X}_{t_n})}\right), k = 1,\ldots,N,$$

with the convention that $\pi_k = \pi_k(TN^{-1})$. Then

- $$E^{h,X_0}[e(N)] = \sum_{n=0}^{N-1} E^{h,X_0}\left[v(t_{n+1},\widehat{X}_{t_{n+1}})\pi_{n+1} - v(t_n,\widehat{X}_{t_n})\pi_n\right]$$

$$= \sum_{n=0}^{N-1} E^{h,X_0}\pi_n\left(v(t_{n+1},\widehat{X}_{t_{n+1}})\exp\left(T\frac{\sigma^2(\widehat{X}_{t_n})h''(\widehat{X}_{t_n})}{2Nh(\widehat{X}_{t_n})}\right) - v(t_n,\widehat{X}_{t_n})\right)$$

$$v(t_{n+1}, \widehat{X}_{t_{n+1}}) \exp\left(T \frac{\sigma^2(\widehat{X}_{t_n}) h''(\widehat{X}_{t_n})}{2Nh(\widehat{X}_{t_n})}\right) - v(t_n, \widehat{X}_{t_n}) = M + I_1 + I_2 + I_3,$$

where M is a (local) martingale increment,

$$I_1 = \int_{t_n}^{t_{n+1}} \frac{\pi_{n+1}(t - t_n)}{\pi_n(t - t_n)} \frac{\sigma^2(\widehat{X}_{t_n}) v_x(t, \widehat{X}_t) \mu(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} dt$$

and I_1 and I_2 are similarly complicated integrals containing

$$\frac{1}{h(\widehat{X}_t)} \text{ and } \int_{t_n}^{t_{n+1}} \frac{\sigma^2(\widehat{X}_{t_n}) h^{-p}(\widehat{X}_t)}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} dt$$

for some $p \in (0, 3)$.

Computing inverse moments

- Typical approach in the literature towards computing uniform bounds on moments is via Ito's formula and controlling the (local) martingale terms using BDG inequality.

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Computing inverse moments

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- One difficulty with the first approach in the present case is that the local martingale term in the decomposition of, e.g., $h^{-1}(X)$, is a strict local martingale.
- The works of Alfonsi and Neuenkirch & Szpruch study in particular the inverse moments of

$$dY_t = dB_t + f(Y_t)dt$$

for a large class of conservative diffusions in a given interval but their conditions on f cannot be satisfied when $f = \frac{h'}{h}$ with (h, M) being a recurrent transformation, as it implies the Radon-Nikodym density $\frac{dR}{dP}$ is an R-martingale.

A comparison result

- Consider the case $r = \infty$, and define A by $A_0 = 0$ and

$$dA_t = \frac{\sigma^2(\widehat{X}_{t_n})}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)} dt, \qquad t \in (t_n, t_{n+1}].$$

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- Also assume that σ is bounded. Thus, $A_t \leq t\|\sigma^2\|_\infty$.
- Set $\widehat{Y}_t = \widehat{X}_{A_t^{-1}}$ and recall (11). DD-S Theorem yields

$$d\widehat{Y}_t = d\beta_t + \left(\frac{h'}{h}(\widehat{Y}_t) + \mu_t\right) dt,$$

for some μ_t with $\mu_t \geq c_1$, where β is $(\mathcal{F}_{A_t^{-1}})$-Brownian motion.
- By comparison, for any non-increasing ϕ,

$$E^{h,X_0}(\phi(\widehat{X}_t)) \leq E^{h,X_0}(\phi(Y_{A_t})),$$

where

$$Y_t = X_0 + \beta_t + \int_0^t \left(\frac{h'}{h}(Y_s) + c_1\right) ds. \qquad (12)$$

Inverse moments

- Since h is increasing when $r = \infty$, the above in particular allows us to bound $E^{h,X_0}(\frac{1}{h}(\widehat{X}_t))$, uniformly in N, via Y.
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Inverse moments

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- A difficulty, however, is that we need the moment of $\frac{1}{h(Y)}$ at a rather arbitrary stopping time.
- The potential theory developed for Schrödinger semigroups comes to our rescue.
- Let's allow again r to be finite and consider

$$dY_t = dW_t + \left\{ \frac{h'(Y_t)}{h(Y_t)} + c \right\} dt, \quad t < \zeta(Y), \tag{13}$$

where $c \leq 0$ if $r=\infty$ and is unconstrained otherwise. $\zeta(Y)$ above denotes the first time that Y exits (ℓ, r).

The second key lemma

Let Y be the process defined by (13) with $Y_0 = X_0$. Then the following statements are valid:

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$$E^{h,X_0}\left[\frac{1}{h(Y_S)}\right] < \frac{K}{h(X_0)}.$$

3. For any $t > 0$ and $p \in [0,1)$

$$E^{h,X_0}\left[\int_0^t \frac{1}{h^{2+p}(Y_s)}ds\right] < \infty.$$

Some moment estimates for the BEM scheme

Suppose h satisfies the conditions of the first key lemma and σ is bounded. Let $T > 0$, $p \in [0, 1)$ and $t(s) = s - t_n$. Then

1.
$$\sup_{\substack{N,\\ t\leq T}} E^{h,X_0}\left(\frac{1}{h}(\widehat{X}_t) + \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \frac{\sigma^2(\widehat{X}_{t_n})h^{-2-p}(\widehat{X}_t)}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)}dt + |\widehat{X}_t|^m\right) < \infty$$

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2 Let $p \in [0,1)$ and $m \geq 0$ be an integer. For each n

$$E^{h,X_0}\left(\int_{t_n}^{t_{n+1}} \left|\frac{h^{1-p}(\widehat{X}_t)(1+\widehat{X}_t^m)\mu(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}\right| dt \Big| \mathcal{F}_n\right)$$

$$\leq \frac{KT}{N} E^{h,X_0}\left(\int_{t_n}^{t_{n+1}} \frac{\sigma^2(\widehat{X}_{t_n})(h^{-2-p}(\widehat{X}_t)+\widehat{X}_t^m)}{H_x^2(t_n,\widehat{X}_{t_n};t,\widehat{X}_t)}dt \Big| \mathcal{F}_n\right).$$

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$$\leq \frac{KT}{N}E^{h,X_0}\left(\int_{t_n}^{t_{n+1}}\frac{\sigma^2(\widehat{X}_{t_n})(h^{-2-p}(\widehat{X}_t)+\widehat{X}_t^m)}{H_x^2(t_n, \widehat{X}_{t_n}; t, \widehat{X}_t)}dt\Big|\mathcal{F}_n\right).$$

3. If $p \leq \frac{1}{2}$ and $\frac{h''}{h^{1-p}}$ is bounded, denoting $\sigma(\widehat{X}_{t_n})$ by σ_n,
$$E^{h,X_0}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\left\{1 - e^{t(s)\sigma_n^2\frac{h''}{2h}(\widehat{X}_{t_n})}\right\}\frac{\sigma_n^2(h^{-p}(\widehat{X}_s)+\widehat{X}_s^m)}{H_x^2(t_n, \widehat{X}_{t_n}; s, \widehat{X}_s)}ds < \frac{KT}{N}.$$

Relevant PDE estimates

Suppose $\sigma \in C_b^4((0,r)$, $g \in C_b^6((0,r),\mathbb{R})$ with $g^{(k)}(0) = 0$ (and $g^{(k)}(r) = 0$ if $r < \infty$) for $k \in \{0,1,2,3,4\}$,

$$\frac{|h^{(k)}|}{h} < \frac{K_h}{h^{k-2+p}}, \qquad k \in \{2,3,4\},$$

for some K_h and $p \in (0,1)$, and recall v from (9).

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for some K_h and $p \in (0,1)$, and recall v from (9). Then,

$$v_t + \frac{\sigma^2}{2} v_{xx} + \sigma^2 \frac{h'}{h} v_x = -\sigma^2 v \frac{h''}{2h}. \tag{14}$$

Moreover, v and v_t are uniformly bounded and there exists a constant K such that

$$\sup_{t \le T} \left| \frac{\partial^k}{\partial x^k} v_t(t,x) \right| + \sup_{t \le T} \left| \frac{\partial^k}{\partial x^k} v(t,x) \right| \le K h^{2-p-k}(x), \qquad k \in \{1,2\}. \tag{15}$$

Hypotheses for the weak convergence estimates

Assumption 1

The functions σ, h and g satisfy the following regularity conditions.

1. $h \in C^4((0,r),(0,\infty))$ *such that*

$$\frac{|h^{(k)}|}{h} < \frac{K_h}{h^{p+k-2}}, \qquad k \in \{2,3,4\},$$

   for some K_h *and* $p \in [0, \frac{1}{2}]$.

2. $\sigma \in C_b^2((0,r),(0,\infty))$ *is bounded away from* 0.
3. g *is of polynomial growth with* $g(0) = 0$ *(*$g(r) = 0$ *if* $r < \infty$*).*
4. $v \in C^{1,4}((0,r),\mathbb{R})$*, satisfies (14) and for* $k \in \{1,2\}$

$$\sup_{t \le T} \left| \frac{\partial^k}{\partial x^k} v_t(t,x) \right| + \sup_{t \le T} \left| \frac{\partial^k}{\partial x^k} v(t,x) \right| \le K(1 + x^m) h^{2-p-k}(x),$$

   for some constant K *and integer* $m \ge 0$.

Back to convergence rate estimates

Under Assumption 1,

$$\left| E^{h,X_0}[I_1 + I_2 + I_3 | \mathcal{F}_n] \right|$$

$$\leq K \frac{T}{N} E^{h,X_0} \left(\int_{t_n}^{t_{n+1}} \frac{\sigma^2(\widehat{X}_{t_n})(h^{-2-p}(\widehat{X}_s) + \widehat{X}_s^m)}{H_x^2(t_n, \widehat{X}_{t_n}; s, \widehat{X}_s)} ds \Big| \mathcal{F}_n \right)$$

$$+ E^{h,X_0} \left(\int_{t_n}^{t_{n+1}} \left(1 - \exp\left(t(s)\sigma_n^2 \frac{h''}{2h}(\widehat{X}_{t_n}) \right) \right) \frac{\sigma_n^2(h^{-p}(\widehat{X}_s) + \widehat{X}_s^m)}{H_x^2(t_n, \widehat{X}_{t_n}; s, \widehat{X}_s)} ds \Big| \mathcal{F}_n \right)$$

$$+ K \frac{T}{N} E^{h,X_0} \left(\int_{t_n}^{t_{n+1}} \frac{\sigma(\widehat{X}_{t_n})^2(h^{-2}(\widehat{X}_s) + \widehat{X}_s^m)}{H_x^2(t_n, \widehat{X}_{t_n}; s, \widehat{X}_s)} ds \Big| \mathcal{F}_n \right) \leq K \frac{T}{N}$$

Go to Moment Estimates

Numerical experiments

Numerical pricing of barrier options

- We shall apply our scheme to a down-and-out option in the Black-Scholes model and a double barrier option in hyperbolic local volatility model, where the local volatility is given by

$$\sigma(x) = \nu\left\{\frac{(1-\beta+\beta^2)}{\beta}x + \frac{(\beta-1)}{\beta}\left(\sqrt{x^2+\beta^2(1-x)^2}-\beta\right)\right\}.$$

- To achieve σ away from zero on (ℓ, r), we shall consider log price in the Black-Scholes model.
- $h(x) = e^{-\ell} - e^{-x}$ in the one sided case whereas $h(x) = (x-\ell)(r-x)$ in the double barrier case. Neither h satisfies the condition of Assumption 1.

Single barrier put in Black-Scholes

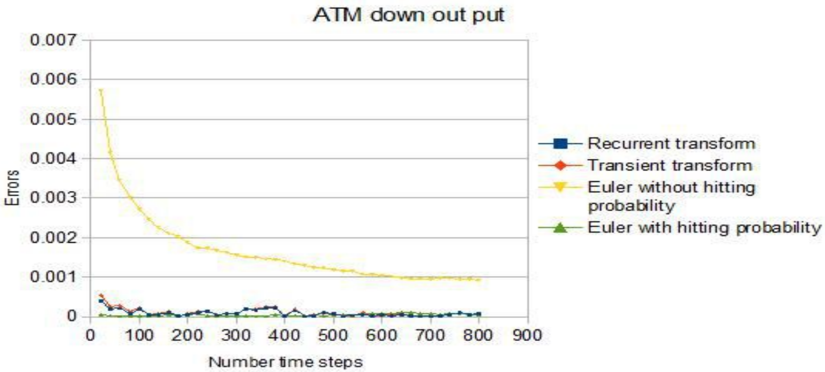

Figure: Absolute discrepancy between the benchmark price for ATM down-and-out put and those calculated with different numerical schemes when $S_0 = 1$, $K = 1$, $T = 1$ year, $l = \log(b = 0.8)$, $r = +\infty$ and $\sigma = 20\%$.

ATM double barrier call in HLV

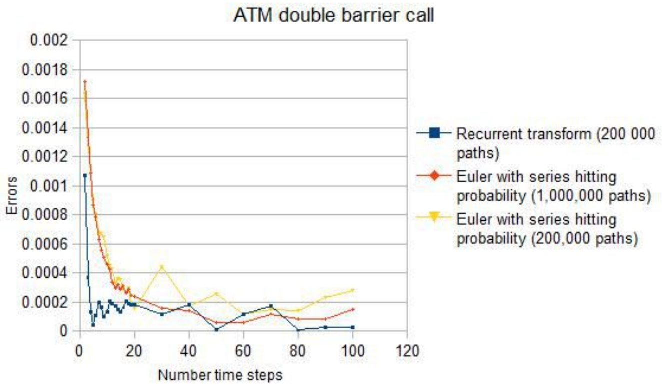

Figure: Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ATM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.85$, $B = 1.25$.

OTM double barrier call in HLV

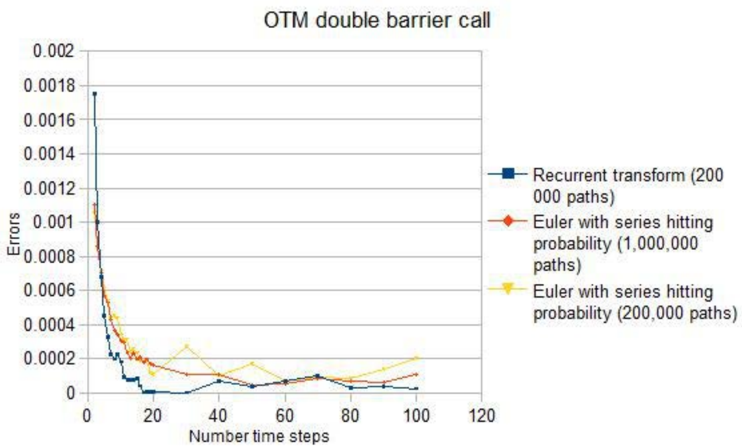

Figure: Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for OTM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.85$, $B = 1.25$.

ITM double barrier call in HLV

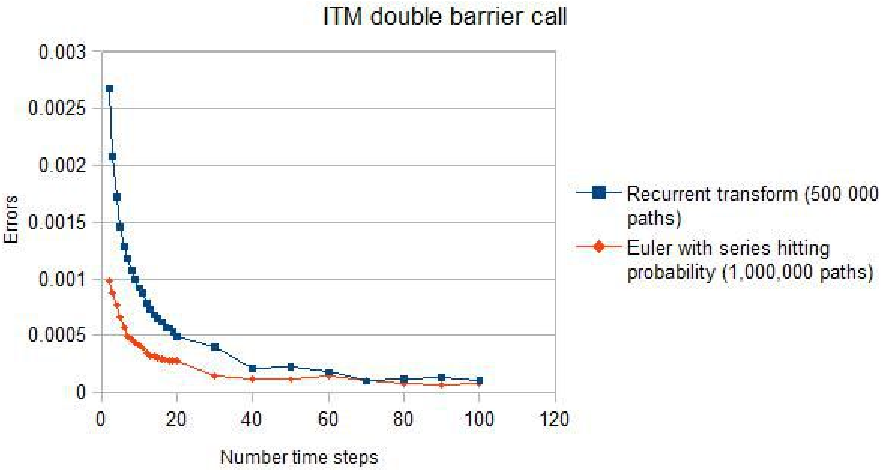

Figure: Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ITM double barrier call when $S_0 = 1$, $K = 0.9$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.15$.

ATM double barrier call in HLV

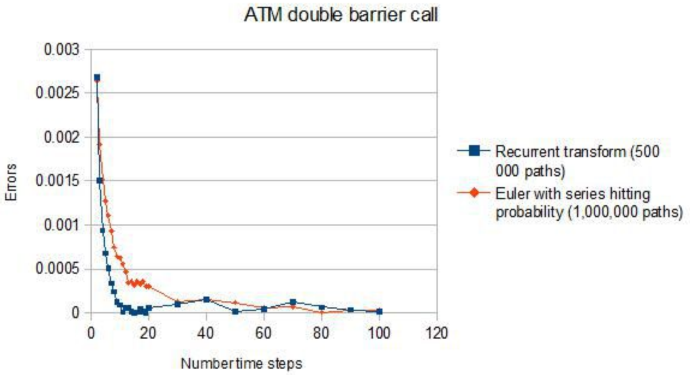

Figure: Absolute discrepancy between the benchmark price and those calculated by different numerical schemes for ATM double barrier call when $S_0 = 1$, $K = 1$, $\nu = 20\%$, $\beta = 0.5$, $T = 1$ year, $b = 0.8$, $B = 1.3$.

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- Introduced a novel drift-implicit scheme for killed diffusions that brings the weak convergence rate back to $O(1/N)$.
- Moment estimates are calculated using potential theory.
- The earlier drift-implicit works that rely on BDG type inequalities for moment estimates impose restrictions on h'/h, which in turn imply $\frac{1}{h(X)}\exp(\frac{1}{2}\int_0 \frac{f(X_s)}{h(X_s)}ds)$ is a $R^{h,x}$-martingale. This is not possible.
- Numerical experiments are consistent with theoretical results despite h not satisfying the stated conditions.

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