SPDEs in finance and their statistical inference

Igor Cialenco
Department of Applied Mathematics
Illinois Institute of Technology
cialenco@iit.edu ◇ ◇ ◇ http://cialenco.com

Bachelier Finance Society One World Seminar

October 26, 2023

A stochastic partial differential equation (SPDE)

=

PDE + random noise

$$\mathrm{d}u(t,x) - \theta u_{xx}(t,x)\,\mathrm{d}t = \sigma\,\mathrm{d}W(t,x), x \in [0,1], t \geq 0.$$

Adding noise may come from:

- ◇ passing to the limit from microscopic to mesoscopic/macroscopic level;
- ◇ model misspecification, i.e. a PDE is an approximated/ideal model;
- ◇ byproduct of other problems; e.g. filtering;
- ◇ reasonable modeling tool; e.g. finance, economics;
- ◇ other.

Note: In our terms, SPDEs $\neq$ 'random PDEs', where usually the focus is on random coefficients, random initial or/and boundary data.

Heath, Jarrow, Morton (HJM) model [HJM92] model for the instantaneous forward rate $f(t,T) = -\partial \log P(t,T)/\partial T$. Using Musiela parametrization $f(t,x) := f(t,t+x)$, it reads

$$\mathrm{d}f(t,x) = \left(\frac{\partial}{\partial x} f(t,x) + \alpha(t,x)\right)\mathrm{d}t + \sum_{j=1}^{\infty} \sigma^i(t,x)\,\mathrm{d}w^i(t),$$

where w^i are independent 1D Brownian motions, and α given by no-arbitrage condition

$$\alpha(t,x) = \sum_{i=1}^{\infty} \sigma(t,x)\left(\int_0^{\infty} \sigma^i(t,x)\,\mathrm{d}u + \lambda^i(t)\right).$$

Later Brace, Gatarek, Musiela (BGM) [BGM97, Bra08]; see also [Kus00, BSC^{+}99, SCS01], and Filipovic [Fil01].

Using PCA approach Cont [Con05], proposes to decompose

$$f(t,x) = r(t) + s(t)[y(x) + u(t,x)],$$

where r is the short rate, s the spread, y a deterministic function, and u is the deformation process, and postulated to follow the dynamics

$$\mathrm{d}u(t,x) = (\kappa u_{xx}(t,x) + u_x(t,x))\,\mathrm{d}t + \sigma W(t,x), \quad x \in [0,1], t \geq 0,$$

where $\kappa, \sigma > 0$ model parameters. A parabolic SPDE.

See also [BK01] for SPDEs in pricing interest rate derivatives.

More details on SPDEs in interest rates models in [CT06].

SPDEs in Finance: forward utility measures

Musiela and Zariphopoulou [MZ10a] introduce the notion of forward utilities, in the context of optimal investment.

Assume that X_t^π is the wealth process corresponding to a self-financing admissible trading strategy π, in a standard Ito diffusion type market model. A forward utility measures $U(t,x)$ is a function such that the optimal investment is consistent across all times, $U(t, X_t^\pi)$ is super-martingale for any admissible $\pi \in \mathcal{A}$, and martingale for the optimal strategy π^*.

Then, U solves the following (nonlinear) SPDE

$$\mathrm{d}U(t,x) = \frac{1}{2} \frac{|U_x(t,x)\lambda_t + \sigma_t \sigma_t^+ a(t,x)|^2}{U_{xx}(t,x)} + a(t,x)\,\mathrm{d}W_t$$

See also [MZ10b, KOZ18, NZ19], and many more.

SPDEs in Finance: Limit Order Book

Cont and Muller [CM21]. Let $V(t,p)$ denote the volume in LOB at time t and price p. For a small bid-ask spread and small tick size δ, LOB can be described by a continuum approximation through its density $v(t;p)$ representing the volume of orders per unit price: $V(t;p) \simeq v(t;p)\delta$. With S_t being the mid-price, define the centered order book density

$$u(t,x) = v(t, S_t + x).$$

Following some heuristics derivations, the SPDE for $u(t,x)$ is derived

$$\begin{aligned}
\mathrm{d}u &= [\eta_a \Delta u + \beta_a \nabla u + \alpha_a u + f_a]\,\mathrm{d}t + \sigma_a u\,\mathrm{d}W^a(t), \quad x \in (0,L) \\
\mathrm{d}u &= [\eta_b \Delta u + \beta_b \nabla u + \alpha_b u + f_b]\,\mathrm{d}t + \sigma_b u\,\mathrm{d}W^b(t), \quad x \in (-L,0) \\
&\quad u(t,x) \le 0,\ x<0, \quad u(t,x) \ge 0,\ x>0 \\
&\quad u(t,0+) = u(t,0-) = 0, \quad u(t,-L) = u(t,L) = 0.
\end{aligned}$$

See also Lototsky et al. [LSZ21], Hambly et al. [HKN20].

SPDEs in Finance: other topics

- ♢ Implied volatility surface: Brace et al. [BFG07]
- ♢ Pricing of mortgage backed securities (MBS): Ahmad, Hambly, Ledger [AHL18]
- ♢ Large portfolio, propagation of chaos: Hambly and Kolliopoulos [HK17], and Kolli and Shkolnikov [KS18]
- ♢ Systemic risk: Hambly and Soejmark [HS19]
- ♢ DNN passing to the limit
- ♢ ...

Consider the stochastic heat equation

$$\begin{cases} \mathrm{d}u(t,x) - \theta u_{xx}(t,x)\,\mathrm{d}t = \sigma\,\mathrm{d}W(t,x), \\ u(0,x) = u_0(x), \end{cases}$$

for $x \in G \subseteq \mathbb{R}^d, t \geq 0$, and where σ and θ are positive parameters of interest, and W is a cylindrical Brownian motion.

Observations:

Assume that we observe one path of the solution (or a projection of the solution) discretely or continuously in time and/or space.

Goal:

Find consistent and asymptotically normal estimators for θ and/or σ.

Why:

model estimation; model validation; . . .

$$\mathrm{d}u(t,x) - \theta u_{xx}(t,x)\,\mathrm{d}t = \sigma\,\mathrm{d}W(t,x), \quad x \in [0,\pi]$$

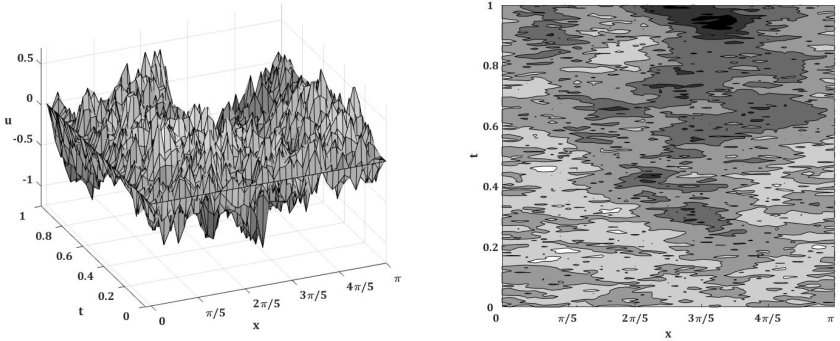


Figure: *The observable: a sample path of the solution*

Goal: Find or estimate θ and/or σ

$dX(t) = \theta X(t)dt + \sigma X(t)dW(t), \quad t \geq 0$ under probability $\mathbb{P}$

`Assume we observe one path of the solution, continuously in time.`

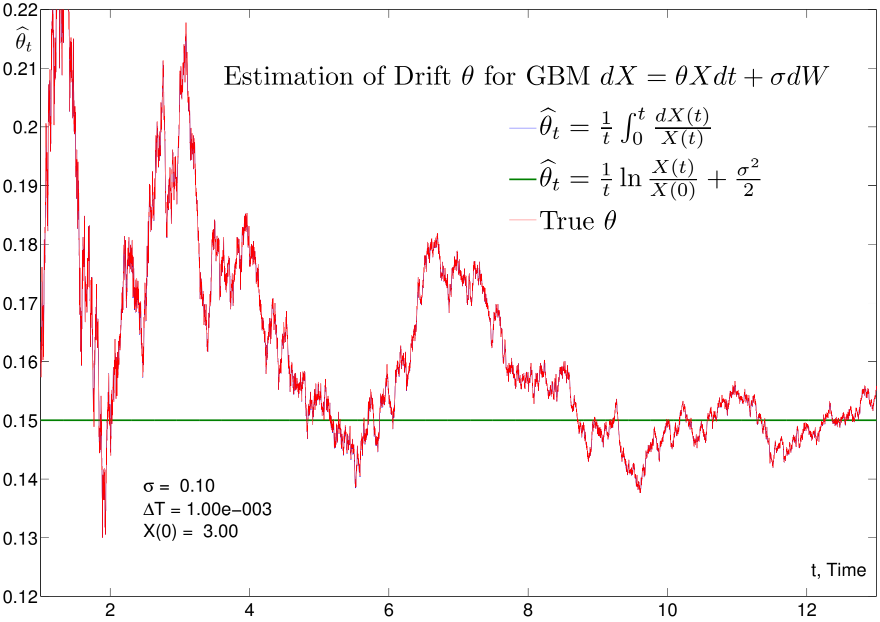
Girsanov Theorem (change of drift/ find likelihood ratio):
Under some *"technical conditions"*, $\exists$ a probability measure $\mathbb{P}_0 \sim \mathbb{P}$, such that $dX_t = \sigma X_t dB_t$, under $\mathbb{P}_0$, and

$$\frac{d\mathbb{P}}{d\mathbb{P}_0} = \exp\left(\frac{\theta}{\sigma^2} \int_0^t \frac{dX_t}{X_t} - \frac{\theta^2 t}{2\sigma^2} \right).$$

$d\mathbb{P}/d\mathbb{P}_0$ - the Likelihood Ratio (Radon-Nikodym derivative).
Maximize Likelihood Ratio $\Rightarrow$ MLE

$$\widehat{\theta}_t = \frac{1}{t}\int_0^t \frac{dX(s)}{X(s)} = \frac{1}{t}\log\frac{X(t)}{X(0)} - \frac{\sigma^2}{2},$$

$$\boxed{\hat{\theta}_t \to \theta\ , \quad t \to \infty}$$



Estimation of Drift θ for GBM $dX = \theta X dt + \sigma dW$
$\widehat{\theta}_t = \frac{1}{t}\int_0^t \frac{dX(t)}{X(t)}$
$\widehat{\theta}_t = \frac{1}{t}\ln\frac{X(t)}{X(0)} + \frac{\sigma^2}{2}$
True θ
$\widehat{\theta}_t$
σ = 0.10
ΔT = 1.00e−003
X(0) = 3.00
t, Time
0.22
0.21
0.2
0.19
0.18
0.17
0.16
0.15
0.14
0.13
0.12
2
4
6
8
10
12

Intro: estimating volatility σ in ODEs

$$dX(t) = \theta X(t)dt + \sigma X(t)dW(t), \quad t \geq 0.$$

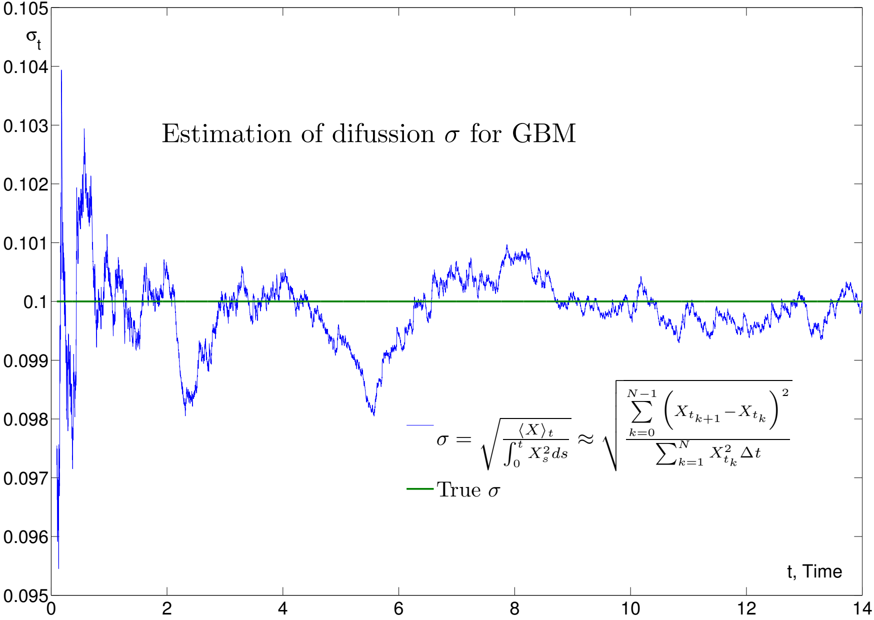
Using Quadratic Variation argument:

$$\langle X \rangle_T = \sigma^2 \int_0^T X_s^2 ds.$$

Hence,

$$\sigma = \sqrt{\frac{\langle X \rangle_T}{\int_0^T X_s^2 ds}} \approx \sqrt{\frac{\sum\limits_{k=0}^{N-1} \left(X_{t_{k+1}} - X_{t_k} \right)^2}{\sum_{k=1}^{N} X_{t_k}^2 \Delta t}} \to \sigma, \quad \Delta t \to 0,$$

with $0 = t_0 < t_1 < \dots t_N = T$, for some fixed T.


Estimation of difussion σ for GBM
σ_t
t, Time
$\sigma = \sqrt{\frac{\langle X \rangle_t}{\int_0^t X_s^2 ds}} \approx \sqrt{\frac{\sum_{k=0}^{N-1} \left(X_{t_{k+1}} - X_{t_k} \right)^2}{\sum_{k=1}^{N} X_{t_k}^2 \Delta t}}$
True σ
0.105
0.104
0.103
0.102
0.101
0.1
0.099
0.098
0.097
0.096
0.095
0
2
4
6
8
10
12
14

Stochastic ODE: conclusion

For observations $X(t),\ t \in [0,T]$,

the drift θ - approximated, **the volatility σ - exactly**

Stochastic ODE: conclusion

For observations $X(t),\ t \in [0,T]$,

the drift θ - approximated, **the volatility σ - exactly**

WHY?

Stochastic ODE: conclusion

For observations $X(t),\ t \in [0, T]$,
the drift θ - approximated, **the volatility σ - exactly**

WHY?

◇ `Regular model`
1) $\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}$ exists; 2) has a special form (LAN)
Same statistical estimation procedure for all models
Find MLE by maximizing likelihood ratio

◇ `Singular model` otherwise
Individual approach
In particular, if $\mathbb{P}_{\sigma_1} \perp \mathbb{P}_{\sigma_2}$ for $\sigma_1 \neq \sigma_2$, then one may find σ exactly

`What do we have for SPDEs?`

Stochastic ODE: conclusion

For observations $X(t),\ t \in [0,T]$,
the drift θ - approximated, **the volatility σ - exactly**

WHY?

◇ `Regular model`
1) $\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}$ exists; 2) has a special form (LAN)
Same statistical estimation procedure for all models
Find MLE by maximizing likelihood ratio

◇ `Singular model` otherwise
Individual approach
In particular, if $\mathbb{P}_{\sigma_1} \perp \mathbb{P}_{\sigma_2}$ for $\sigma_1 \neq \sigma_2$, then one may find σ exactly

`What do we have for SPDEs?`
Mostly singular, hence individual approaches are needed.

Statistics for SPDEs: main methods

- Spectral approach:
  sampling in the Fourier space in a continuous time setup.
- Sampling in physical domain:
  discrete sampling in time and/or space.
- Local measurements:
  sample the solution locality in space and continuously in time.

For up to date bibliography, a dedicated web-site:
https://sites.google.com/view/stats4spdes/

Part II:

Spectral Approach

Spectral Approach

introduced by M. Huebner, R. Khasminskii, B. Rozovskii [HKR93, HR95]
Consider, the stochasrtic heat eq. with additive noise,

$$\mathrm{d}u(t,x) = \theta u_{xx}(t,x)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t,x),$$

zero initial data, Dirichlet b.c., $x \in [0,\pi]$, $\sigma > 0$ known, θ unknown.

The noise:

- ♢ the (negative) Laplace operator $-\Delta u = -u_{xx}$ has (only) a discrete spectrum $\nu_k = k^2, k \in \mathbb{N}$.
- ♢ its eigenfunctions $h_k(x) = \sqrt{2/\pi}\sin(kx)$ form a CONS in L^2.
- ♢ The noise term can be written as

$$\mathrm{d}W(t,x) = \sum_{k=1}^{\infty} h_k(x)\,\mathrm{d}w_k(t),$$

where $w_k(t)$ are independent standard 1D Brownian motions.

The solution can be written as

$$u(t,x) = \sum_{k=1}^{\infty} h_k(x) u_k(t),$$

where $u_k(t) = (u, h_k)_{L^2}$ are the Fourier coefficients/modes. Clearly

$$\mathrm{d}u_k(t) + \theta k^2 u_k \,\mathrm{d}t = \sigma \,\mathrm{d}w_k(t), \quad k \geq 1.$$

Let $H^N = \mathrm{Span}\{h_1, \ldots, h_N\}$, and P^N the projection of $H = L^2$ on H^N. Respectively, we put

$$u^N = P^N u \simeq (u_1, \ldots, u_N).$$

Note that u^N follows the dynamics of a finite dimensional system of decoupled SODEs.

The observations: Assume that we observe one path of the first N Fourier modes continuously over a finite interval of time $[0, T]$, i.e. we observe/measure

$$u^N(t) = (u_1(t), \dots, u_N(t)), \quad t \in [0, T]$$

for one $\omega \in \Omega$.

Possible asymptotic regimes

- ♢ Large times $T \to \infty$
- ✓ Large number of Fourier modes (fine space) $N \to \infty$
- ♢ Small noise $\sigma \to 0$
- ♢ Combinations of the above

Denote by $\mathbb{P}_\theta(A) = \mathbb{P}(u^N \in A),\ A \in \mathcal{B}(C[0, T])$, the measure generated by the solution u^N.

These are 'diagonalizable models', in the Fourier space.

Usually, $\mathbb{P}_{\theta_1}(u^N) \sim \mathbb{P}_{\theta_2}(u^N)$. It is a finite dimensional system of SODEs.

The likelihood ratio (the Radon–Nikodym derivative), $\frac{\mathrm{d}\mathbb{P}_{\theta_1}(u^N)}{\mathrm{d}\mathbb{P}_{\theta_0}(u^N)}$ is computed by Girsanov theorem. There exists Maximum Likelohood Estimator (MLE)

$$
\begin{aligned}
\widehat{\theta}_N &= \arg\max_{\theta_1} \ \log \frac{\mathrm{d}\mathbb{P}_{\theta_1}(u^N)}{\mathrm{d}\mathbb{P}_{\theta_0}} \\
&= -\frac{\sum_{n=1}^{N} \int_0^T k^2 u_k(t)\,\mathrm{d}u_k(t)}{\sum_{k=1}^{N} \int_0^T k^4 u_k^2\,\mathrm{d}t}.
\end{aligned}
$$

consistency & asymptotic normality

$$
\widehat{\theta}_N \xrightarrow[N\to\infty]{\text{a.s.}} \theta, \qquad N^{\frac{3}{2}}(\widehat{\theta}_N - \theta) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, c\sigma/T).
$$

… since than … see survey [Cia18]

- Bayesian: Bishwal ('02), Cheng, IgC, Gong ('18)
- Several parameters: Huebner ('97)
- $\theta(t)$-random: Lototsky ('04)
- Small noise: Huebner ('97), Ibragimov-Khasminskii ('98,'99)
- 'almost' diagonalizable: Rozovskii-Lototsky ('97, '01)
- Additive fractional noise: IgC, Lototsky, Pospisil ('09)
- Multiplicative noise: IgC and Lototsky ('08), IgC ('10)
- Hypothesis testing: IgC and Xu ('14, '15)
- Trajectory fitting estimators: IgC, Gong, Huang ('16)
- **Nonlinear SPDE**: IgC and Glatt-Holtz ('11) 2D Navier-Stocks, semilinear SPDEs [PS20], IgC, Ruimeng Hu, Quyuan Lin [CHL23] Stochastic Primitive Equations.

Spectral approach for nonlinear SPDEs; main ideas

On a given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, consider the evolution equation

$$dU(t) = \theta A U(t)dt + F(U)dt + \sigma dW(t), \quad U(0) = U_0.$$

- ♢ assume that $U(\omega, t)$ belongs to some "suitable" Hilbert space $\mathcal{H}$; in particular $U = U(\omega, t, x)$
- ♢ $(-A)$ a linear, selfadjoint, positive-defined (think (-Laplace)$^\beta$) in $\mathcal{H}$ with eigenfunctions $\{\Phi_k\}_{k\geq 1}$ CONS in $\mathcal{H}$
- ♢ $\sigma dW(t) = \sum_{k\geq 1} \sigma_k \Phi_k dW_k(t)$, $W_k, k \in \mathbb{N}$ ind. Brownian Motions
- ♢ F maybe nonlinear, 'subordinated' to A; σ known
- ♢ U observed for all $t \in [0, T]$ - **continuous observations**

Goal:

Find estimators $\hat{\theta}(\omega)$, $\omega \in \Omega$, for parameters θ by **observing a single outcome** $u = u(\omega, t) \in \mathcal{H}$ over a finite time horizon $t \in [0, T]$.

Formal Procedure to Derive an Estimator

- Project the full system down to N dimensions $P_N(\mathcal{H}) = \mathcal{H}_N \simeq \mathbb{R}^N$

$$dU^N = (\theta A U^N + \Psi_N)dt + P_N \sigma dW, \quad U(0) = U_0$$

- Let $\mathbb{P}_\theta^{N,T}(\cdot) = \mathbb{P}(U^N \in \cdot)$ be the measure on $C([0,T];\mathbb{R}^N)$ generated by U^N;
  $\mathbb{P}_\theta^T$ be the measure generated by U on $C([0,T];\mathcal{H})$.
- Usually (at least in linear case), we can prove that $\mathbb{P}_{\theta_1}^{N,T} \sim \mathbb{P}_{\theta_2}^{N,T}$
  Hence, get MLE type estimators $\widehat{\theta}_{N,T}$.
- Usually (at least in linear case) $\mathbb{P}_{\theta_1}^T \perp \mathbb{P}_{\theta_2}^T$;
  An indication that the true parameter θ can be found exactly.

Formal Procedure to Derive an Estimator in Nonlinear Case

- Formally treat $\Psi_N = P_N F(U)$ as an external and known quantity (independent of θ)
- Assume that $P_N\sigma$ is invertible on H_N
- Take $G := P_N\sigma(U)(P_N\sigma(U))^*$ and assume it commutes with A
- For a reference values θ_0, apply (formally) Girsanov Theorem and get the 'Likelihood Ratio' (Radon-Nikodym derivative) $d\mathbb{P}_\theta^{N,T}/d\mathbb{P}_{\theta_0}^{N,T}$
- Maximize the Log-Likelihood Ratio
$\widetilde{\theta}_{N,T}(\omega) := \arg\max_\theta d\mathbb{P}_\theta^{N,T}/d\mathbb{P}_{\theta_0}^{N,T}(\omega)$

$$\frac{d\mathbb{P}_{\theta}^{N,T}}{d\mathbb{P}_{\theta_0}^{N,T}} = \exp\Big[-\int_0^T (\theta-\theta_0)\langle AU^N, GdU^N(t)\rangle - \frac{1}{2}\int_0^T (\theta^2-\theta_0^2)\langle AU^N, GAU^N dt\rangle - \int_0^T (\theta-\theta_0)\langle AU^N, G\psi^N dt\rangle\Big],$$

$$\hat{\theta}_{1,N} = -\frac{\int_0^T AU_N G_N dU_N + \int_0^T AU_N G_N P_N F(U))dt}{\int_0^T AU_N G_N AU_N dt},$$

$$\hat{\theta}_{2,N} = -\frac{\int_0^T AU_N G_N dU_N + \int_0^T AU_N G_N P_N F(U_N))dt}{\int_0^T AU_N G_N AU_N dt},$$

$$\hat{\theta}_{3,N} = -\frac{\int_0^T AU_N G_N dU_N}{\int_0^T AU_N G_N AU_N dt}.$$

Idea of the proof

Easy to represent:

$$\hat{\theta}_{2,N} = \theta + \frac{\int_0^T \langle AU^N, G\sum_{j=1}^N \sigma_j(U)\Phi_j dW_j(t)\rangle}{\int_0^T AU_N G_N AU_N dt} + \frac{\int_0^T \langle AU^N, G(F^N(U) - F^N(U^N))\rangle dt}{\int_0^T AU_N G_N AU_N dt}$$

$$\hat{\theta}_{3,N} = \theta + \frac{\int_0^T \langle AU^N, G\sum_{j=1}^N \sigma_j(U)\Phi_j dW_j(t)\rangle}{\int_0^T AU_N G_N AU_N dt} + \frac{\int_0^T \langle AU^N, GF^N(U^N)\rangle dt}{\int_0^T AU_N G_N AU_N dt}$$

- ♢ Show that each of 'the excess term converges to zero'
- ♢ Use LLG and CLT for martingales
- ♢ sharp control of the moments of relevant parts
- ♢ 'splitting argument' to deal with the nonlinear part

Splitting argument

Decompose $U = \bar{U} + R =$ linear + nonlinear

$$\mathrm{d}\bar{U} = \theta A U \,\mathrm{d}t + \sigma dW, \quad \bar{U}(0) = \bar{U}_0$$
$$\mathrm{d}R = \theta A R \,\mathrm{d}t + F(U)\,\mathrm{d}t, \quad R(0) = R_0.$$

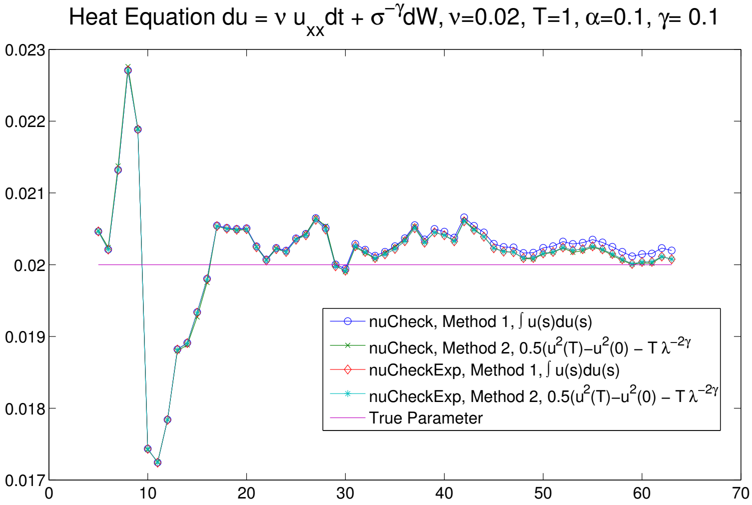
- ◇ Find explicit and exact rates for the moments of the linear part
- ◇ Show that R is slightly 'more regular' than $\bar{U}$, and use this to show that the terms involving R vanish.

Theorem

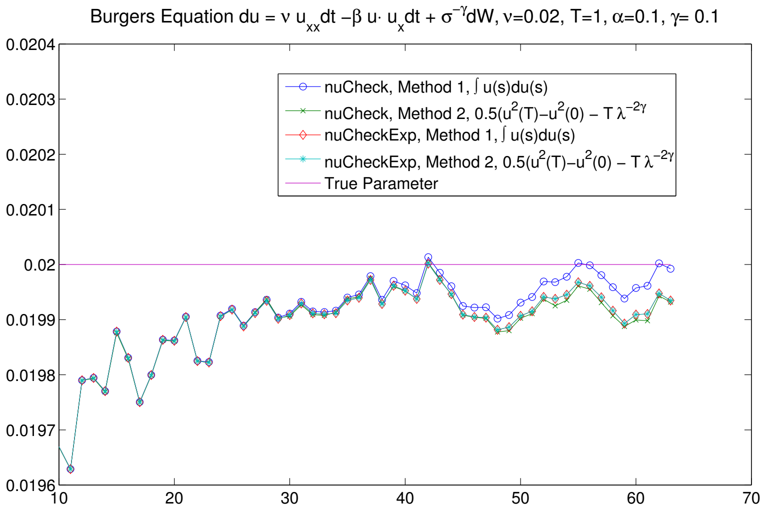
All three estimators $\widehat{\theta}_{1,N}, \widehat{\theta}_{2,N}, \widehat{\theta}_{3,N}$, are consistent and asymptotically normal,

$$\widehat{\theta}_{k,N} \xrightarrow[N\to\infty]{a.s.} \theta, \qquad N^J(\widehat{\theta}_{k,N} - \theta) \xrightarrow[N\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^*),$$

for some known rate of convergence J, depending on the dimension of the space and order of A, and with a known asymptotic variance σ^*.



$$du = \nu u_{xx} dt + dW(t,x), \;\; u(0,x) = u_0, \;\; u(t,0) = u(t,1) = 0$$



$$du = \nu u_{xx} dt - u u_x dt + dW(t,x), \ u(0,x) = u_0, \ u(t,0) = u(t,1) = 0$$

Part III:

Discrete sampled SPDEs in physical domain

Stats for SPDEs: discrete sampling in physical domain

The parameter estimation problem for (linear) SPDEs when the solution is discretely sampled in space and/or time component was addressed systematically only recently by quite different methods: [CH20, BT20] and consequently in [BT19, Cho20, CDVK20, Cho19, KU21, KT19a, KT19b, HT21, CK22, SST20, CKP23], and to [PR97, PT07] for earlier studies, as well as the recent work [HT23] on reaction-diffusion equations.

`Our approach:` use the exact order of regularity and correctly chosen power variation.

[CKP23] C., H.-J. Kim and G. Pasemann *Statistical analysis of discretely sampled semilinear SPDEs: a power variation approach*, forthcoming in Stochastics and Partial Differential Equations: Analysis and Computations, 2023+.

[CK22] C. and H.-J. Kim, *Parameter estimation for discretely sampled stochastic heat equation driven by space-only noise revised*, SPA, 22, pp. 1-30, 2022.

[CH20] C. and Y. Huang, *A note on parameter estimation for discretely sampled SPDEs* Stochastics and Dynamics 20(3), pp. 2050016, 2020 (28 pages, preprint 2017).

The main goal

Statistical analysis of discretely sampled (SPDEs) of the form

$$\mathrm{d}X_t(x) = -\theta(-\Delta)^{\alpha/2} X_t(x)\mathrm{d}t + F(X_t(x))\,\mathrm{d}t + \sigma(-\Delta)^{-\gamma}\mathrm{d}W_t(x),$$

for $x \in [0,1],\ t > 0$, Dirichlet boundary conditions and zero initial data, and where

- $\alpha > 0, \gamma \geq 0$ are given constants,
- W is a cylindrical Wiener process on $L^2([0,1])$,
- F is a (nonlinear) operator acting on some appropriate Hilbert space,
- θ, σ are the parameters of interest (unknown).

Simplest nontrivial example: Stochastic heat equation

$$\mathrm{d}u(t,x) = \theta u_{xx}(t,x)\,\mathrm{d}t + \sigma \sum_{k\geq 1} k^{-2\gamma} \sin(k\pi x)\,\mathrm{d}w_k(t), \quad x \in [0,1],$$

observed at some discrete points (t_k, x_j).

What is the problem?

$$\mathrm{d}X_t(x) = -\theta(-\Delta)^{\alpha/2}X_t(x)\mathrm{d}t + F(X_t(x))\,\mathrm{d}t + \sigma(-\Delta)^{-\gamma}\mathrm{d}W_t(x)$$

- ◇ We know how to treat the case of non-smooth path, both in time and space; for $\gamma = 0$, $X \in C^{1/4-,1/2-}_{t,x}$ [CH20].
- ◇ It is enough to sample discretely the solution $X(t,x)$ in space and/or time at one time point, or one space point, or on a space-time mesh.

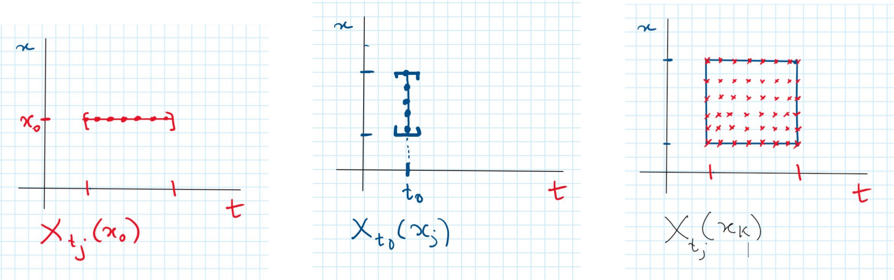


Figure: *Sampling schemes*

$$\mathrm{d}X_t(x) = -\theta(-\Delta)^{\alpha/2}X_t(x)\mathrm{d}t + F(X_t(x))\,\mathrm{d}t + \sigma(-\Delta)^{-\gamma}\mathrm{d}W_t(x)$$

- Larger γ gives smoother solutions.
- Regularity in t can't get better than Hölder 1/2-, and the existing power variation methods apply.

- $X(t,\cdot)$ can reach any order of smoothness, when $\gamma \nearrow \infty$. For example, when $F=0$, $X(t,\cdot)$ is Hölder continuous of order $2\gamma + \alpha/2 - 1/2$.
- We focus on sampling the solution X discretely in spatial variable x and for a fixed $t>0$.

Main line of reasoning

- Take the maximal number of (classical) derivatives in x, say

$$m := \lfloor 2\gamma + \alpha/2 - 1/2 \rfloor$$

- Expect that

$$\partial_x^m X_t(x) \sim \mathrm{fBM}^H + \text{`smooth process'},$$

with $H = 2\gamma + \alpha/2 - 1/2 - m$.

- Adapt the existing results on power variations from [CH20, KT19a, KT19b].
- `It works!`

Main line of reasoning

- Take the maximal number of (classical) derivatives in x, say

$$m := \lfloor 2\gamma + \alpha/2 - 1/2 \rfloor$$

- Expect that

$$\partial_x^m X_t(x) \sim \mathrm{fBM}^H + \text{`smooth process'},$$

with $H = 2\gamma + \alpha/2 - 1/2 - m$.

- Adapt the existing results on power variations from [CH20, KT19a, KT19b].

- `It works!`

`However`, assuming that the process $\partial_x^m X_t(x),\ x \in (0,1)$ is observed, practically speaking is unrealistic.

Main line of reasoning

- Natural idea: approximate the derivatives $\partial_x^m X_t(x)$ by using the discrete measurements of the solution itself, for example by finite differences.
- It does not work!

  Such approximations typically yield a nontrivial and non-vanishing bias in the estimators; see also [CKL20], [CK22].

Main results

Find the needed adjustments (biases) for the naively approximated estimators and prove consistency and asymptotic normality.

Notations

For real valued measurable function $X_t, t \in \mathbb{R}$, we put

$$JX_t := \int_0^t X_r \, \mathrm{d}r, \quad t \in \mathbb{R},$$
$$\Delta_h X_t := X_{t+h} - X_t, \quad t \in \mathbb{R},\ h > 0.$$

For $M, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we have

$$\Delta_h^M (J^m X_t) = \sum_{k=0}^{M} (-1)^{M-k} \binom{M}{k} J^m X_{t+kh}, \quad t \in \mathbb{R},\ h > 0.$$

$C(\mathbb{R})$ denotes the space of continuous and bounded functions on $\mathbb{R}$, endowed with $\|f\|_\infty := \sup |f|$.

$C^k(\mathbb{R}) := \{f \in C(\mathbb{R}) \ : \ \|f\|_{C^k(\mathbb{R})} := \sum_{j \le k} \|D^j f\|_\infty < \infty\}$, for $k \in \mathbb{N}$, and with D being the differential operator.

- Let $\pi = \{t_0, \dots, t_N\}$ be the uniform partition of size N of the interval $[a, b] \subset [0, T]$
- Put $h := h_N := (b - a)/N = t_{k+1} - t_k, k = 0, \dots, N$.
- For fixed $s > 0$, $q, M, N \in \mathbb{N}$, such that $N > M$, we define

$$V_{q,M,s,N}(X) := \frac{1}{b-a} \sum_{k=0}^{N-M} h \left| \frac{\Delta_h^M X_{t_k}}{h^s} \right|^q .$$

- The **Δ-power variation of order (q, M, s) of process X** is defined as

$$V_{q,M,s}(X) := \mathbb{P} - \lim_{N \to \infty} V_{q,M,s,N}(X),$$

provided that the limit (in probability) exists.

Note that $V_{p,1,1}$ corresponds to the (normalized) power variation of order p.

Theorem

Let $q \geq 1$, $s > 0$, $M \in \mathbb{N}$ with $M > s$. Assume that $X \in \mathcal{C}^s([a,b])$ and for some $\alpha > 0$, $\Sigma \geq 0$, the following limit exists

$$h_N^{-\alpha}\left(V_{q,M,s,N}(X) - V_{q,M,s}(X)\right) \xrightarrow{d} \mathcal{N}(0,\Sigma), \quad as\ N \to \infty,$$

where $\mathcal{N}(0,\Sigma)$ is a Gaussian random variable with mean zero and variance Σ. Then, for any $Y \in C^{s+\eta}([a,b])$ with $\eta > \alpha$, and $M > s + \alpha$,

$$h_N^{-\alpha}\left(V_{q,M,s,N}(X+Y) - V_{q,M,s}(X)\right) \xrightarrow{d} \mathcal{N}(0,\Sigma), \quad as\ N \to \infty.$$

Proof based on equivalence of Hölder-Zygmund spaces and classical Hölder spaces.

The case of fBM

- A fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a centered Gaussian process $B^H = (B^H_t)_{t\in\mathbb{R}}$ such that

$$\mathbb{E}\left(B^H_t B^H_r\right) = \frac{1}{2}\left(|t|^{2H} + |r|^{2H} - |t-r|^{2H}\right), \quad t, r \in \mathbb{R}.$$

- A fBM B^H is H-self-similar process with stationary increments.
- Generally speaking, differences of integrals of fBm are not self-similar in the usual sense.
- We extend the notion of self-similarity to a parameterized family of processes, say $X^{(h)}$, $h > 0$. We say that $X^{(h)}$ is **parameterized s-self-similar** if the law of $(h^{-s} X^{(h)}_{ht})_{t\in\mathbb{R}}$ is independent of $h > 0$.

If $M \geq m$, then $\Delta^M_h J^m B^H$ is parameterized $(m+H)$-self-similar and has stationary increments.

Main result for fBM

Theorem

Let $M > m \geq 0$ and $q \geq 1$ be integers, $s = H + m$, and assume that either of the following assumptions is satisfied:

(i) $M = m + 1$ *and* $0 < H < 3/4$,

(ii) $M \geq m + 2$ *and* $0 < H < 1$.

Then, there exists $\sigma_{q,M,s} > 0$ *such that*

$$\sqrt{N}\left(V_{q,M,s,N}\left(J^m B^H\right) - \tau_q \mu_{M,s}^{q/2}\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma_{q,M,s}^2 \mu_{M,s}^q\right), \quad as\ N \to \infty,$$

where $\tau_q := \mathbb{E}|Z|^q$ *with* $Z \sim \mathcal{N}(0,1)$.
Moreover, if q *is an even number, then* $\sigma_{q,M,s}^2 = \sum_{k=1}^{q} \binom{q}{k}^2 \tau_{q-k}^2 \rho_{k,M,s}^2$,
where $\rho_{q,M,s}^2 := q! \sum_{\ell \in \mathbb{Z}} \left(\rho_{M,s}(\ell)\right)^q$.

Recall that $V_{q,M,s}(X) := \mathbb{P} - \lim_{N\to\infty} \frac{1}{b-a} \sum_{k=0}^{N-M} h \left|\frac{\Delta_h^M X_{t_k}}{h^s}\right|^q$.

Remarks

- The proof is based on a version of Breuer-Major Theorem.
- The limit of $V_{q,M,s,N}(J^m B^H)$ depends through $\mu_{M,s}$ on the regularity s of the process as well as the number of differences M.
- Even for small h it is not possible to approximate the rescaled finite difference operator $h^{-1}\Delta_h$ in the definition of $V_{q,M,s,N}(J^m B^H)$ by a derivative operator without introducing a non–trivial bias, due to the change in M and s.
- The constant $\mu_{M,s}$ can be easily computed.

  If $M = 1$, $m = 0$ and $0 < H < 3/4$, then $\mu_{M,s} = 1$.

  If $M = 2$, $m = 1$ and $H = 1/4$, then $\mu_{M,s} = (\sqrt{2} - 1)\frac{16}{15} \approx 0.44$.

  If $M = 2$, $m = 1$ and $H = 1/2$, then $\mu_{M,s} = 2/3$.

The case of semilinear SPDEs

- We consider SPDEs on $\mathcal{D} = (0,1)$ with zero boundary conditions.
- Set $\Phi_k(x) = \sqrt{2}\sin(k\pi x)$ and $\lambda_k = k^2\pi^2$, $k \in \mathbb{N}$. These are the eigenfunctions and eigenvalues of the Laplacian $\Delta = \partial_{xx}$.
- The set $\{\Phi_k\}_{k\in\mathbb{N}}$ forms an orthonormal basis in $L^2(\mathcal{D})$.
- Put $H^s(\mathcal{D}) := \{u \in L^2 \mid \sum_{k=1}^{\infty} \lambda_k^s (u, \Phi_k)^2 < \infty\}$, for $s \in \mathbb{R}$,

We consider the following semilinear SPDE on $L^2(\mathcal{D})$:

$$\mathrm{d}X_t = \Big(-\theta(-\Delta)^{\alpha/2} X_t + F(X_t)\Big)\,\mathrm{d}t + \sigma(-\Delta)^{-\gamma}\mathrm{d}W_t, \quad X_0 \in L^2(\mathcal{D}),$$

where $\alpha, \theta, \sigma > 0$, W is a cylindrical Wiener process on $L^2(\mathcal{D})$, $\gamma > 1/4 - \alpha/4$, and F is a nonlinear operator.

We assume that the above SPDE is well-posed in $L^2(\mathcal{D})$ in the analytically mild and probabilistically weak sense.

Splitting of the solution argument

We use the splitting of the solution argument (see [CGH11, PS20, ACP20]), by writing $X = \overline{X} + \widetilde{X}$, where

$$\mathrm{d}\overline{X}_t = -\theta(-\Delta)^{\alpha/2}\overline{X}_t\mathrm{d}t + \sigma B\mathrm{d}W_t, \quad \overline{X}_0 = 0,$$
$$\mathrm{d}\widetilde{X}_t = \Big(-\theta(-\Delta)^{\alpha/2}\widetilde{X}_t + F(\overline{X}_t + \widetilde{X}_t)\Big)\,\mathrm{d}t, \quad \widetilde{X}_0 = X_0.$$

The solution to the linear equation can be written as a Fourier series

$$\overline{X}_t = \sigma\int_0^t e^{-\theta(t-r)(-\Delta)^{\alpha/2}}B\mathrm{d}W_r = \sum_{k=1}^{\infty}\left(\sigma\lambda_k^{-\gamma}\int_0^t e^{-\theta(t-r)\lambda_k^{\alpha/2}}\mathrm{d}W_r^{(k)}\right)\Phi_k$$
$$=: \sum_{k=1}^{\infty}\overline{x}_k(t)\Phi_k,$$

Fine continuity properties

Proposition

For any $s < 2\gamma + \alpha/2 - 1/2$, *it holds that* $\overline{X} \in C(0,T;C^s(\mathcal{D}))$.

Hence $\overline{X}$ has up to $\lfloor 2\gamma + \alpha/2 - 1/2 \rfloor =: m$ classical derivatives. We call $s^* = 2\gamma + \alpha/2 - 1/2$ `the optimal regularity`. We assume that $s^* \notin \mathbb{N}$.

Proposition

Assume that there exist $\eta, \epsilon > 0$, $0 \leq s_0 < s^*$, *and a continuous function* $g : [0,\infty) \to [0,\infty)$, *such that for any* $s_0 \leq s < s^*$,

$$\|F(u)\|_{s+\eta-\alpha+\epsilon} \leq g(\|u\|_s),$$

where, as before, $\|\cdot\|_s$ *denotes the Hölder-Zygmund norm. Let* $X \in C(0,T;C^{s_0}(\mathcal{D}))$ *and* $X_0 \in C^{s^*+\eta}(\mathcal{D})$. *Then we have* $\widetilde{X} \in C(0,T;C^{s+\eta}(\mathcal{D}))$, *for any* $0 \leq s < s^*$, *and* $X \in C(0,T;C^s(\mathcal{D}))$.

Examples

The above conditions are satisfied for large classes of SPDEs. Some important examples of the nonlinearity F:

1) *(fractional) Heat equation:* In the case $F = 0$, the SPDE becomes linear, sometimes called fractional heat equation, and the Lip condition is trivially satisfied for any $\eta > 0$.

2) *Reaction-diffusion equation:* Let $F(u)(x) = f(u(x))$, where f is a polynomial function or $f \in C_b^\infty(\mathbb{R})$. Then Lip condition is true for any $0 < \eta < 2$.

3) *Advection-diffusion equation:* Let $F(u) = v\partial_x u$ for a given $v \in C^\infty(\mathcal{D})$. Then Lip condition holds with any $0 < \eta < 1$.

4) If $F = F_1 + F_2$, for some F_1, F_2 that satisfy Lip condition with continuous functions g_1, g_2, then F satisfies Lip condition with $g = g_1 + g_2$.

Theorem

Let $t > 0$, $m \in \mathbb{N}_0$, $0 < H < 1$ such that $m + H = s^* = 2\gamma + \alpha/2 - 1/2$. Let $M, q \in \mathbb{N}$, and assume that either $M = m + 1$ with $H < 1/2$ or $M \geq m + 2$. Suppose that Lip condition holds for some $\eta > 1/2$, and assume that θ is known. Then

$$\widehat{\sigma}_N^{q,M} := \tau_q^{-1}(2\theta/(\nu_H \mu_{M,s^*}))^{q/2} V_{q,M,s^*,N}(X_t)$$

with $\nu_H := -\frac{2}{\pi}\Gamma(-2H)\cos(\pi H)$, is a consistent estimator for σ^q, and for any $\epsilon > 0$,

$$\widehat{\sigma}_N^{q,M} = \sigma^q + o_{\mathbb{P}}(N^{-1/2+\epsilon}).$$

If $s^* \in 1/2 + \mathbb{N}_0$, then also

$$\sqrt{N}\left(\widehat{\sigma}_N^{q,M} - \sigma^q\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^{2q}}{\tau_q^2}\sigma^2_{q,M,s^*}\right), \quad as\ N \to \infty.$$

Similar statement holds true for parameter θ.

Numerical example

Consider the stochastic heat equation

$$\mathrm{d}X_t = \theta\Delta X_t \mathrm{d}t + \sigma(-\Delta)^{-\gamma}\mathrm{d}W_t,$$

with initial condition $X_0 = 0$ on $\mathcal{D} = [0,1]$ with Dirichlet BC.

- ◇ Take true values of the parameters $\theta, \sigma = 1$.
- ◇ The smoothing parameter $\gamma \in \{0.0, 0.375, 0.5, 0.625\}$, which correspond to the regularity level $s^* = 2\gamma + 1/2 \in \{0.5, 1.25, 1.5, 1.75\}$.
- ◇ Simulate the path using the Fourier series decomposition of the solution by taking $N_0 = 1 \times 10^4$ eigenmodes.
- ◇ Eigenmodes are simulated by the Euler implicit scheme with $\delta t = 1 \times 10^{-8}$.
- ◇ The solution is computed at $N_0 + 1$ uniformly spaced spatial grid points with step size $h = 1 \times 10^{-4}$.
- ◇ Assume that the solution X is observed at time $T = 1$ on spatial grid points belonging to the interval $[a, b]$, with $a = 0.2$, $b = 0.8$.
- ◇ Apply the main Theorem, with $q = 2$ and $M = \lceil s^* \rceil + 2$.

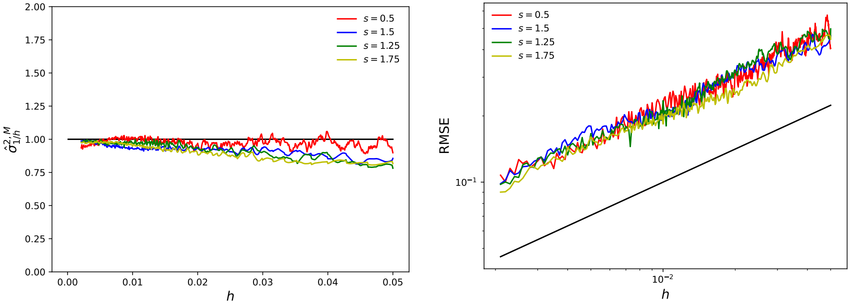


Figure: *Estimation of* σ.

Left panel: the average of 100 *Monte Carlo estimates as function of spatial sampling resolution* h*. The solid black line corresponds to the true value* $\sigma = 1.0$.

Right panel: The RMSE (root mean square error) as function of h*. The black line corresponds to the theoretical convergence rate* $h^{1/2}$.

Thank You !

The end of the talk ...
but not of the story

Bibliography I

R. Altmeyer, I. Cialenco, and G. Pasemann.
Parameter estimation for semilinear SPDEs from local measurements.
Preprint, arXiv: 2004.14728, 2020.

F. Ahmad, B.M. Hambly, and S. Ledger.
A stochastic partial differential equation model for the pricing of mortgage-backed securities.
Stochastic Processes and their Applications, 128(11):3778–3806, nov 2018.

A. Brace, G. Fabbri, and B. Goldys.
An hilbert space approach for a class of arbitrage free implied volatilities models, 2007.

Alan Brace, Dariusz Gatarek, and Marek Musiela.
The market model of interest rate dynamics.
Mathematical Finance, 7(2):127–155, apr 1997.

Bibliography II

Arunabha Bagchi and K. Suresh Kumar.
An infinite factor model for the interest rate derivatives.
In *Mathematical Finance*, pages 59–68. Birkhäuser Basel, 2001.

A. Brace.
Engineering BGM.
Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2008.

Jean-Philippe Bouchaud, Nicolas Sagna, Rama Cont, Nicole El-Karoui, and Marc Potters.
Phenomenology of the interest rate curve.
Applied Mathematical Finance, 6(3):209–232, sep 1999.

Bibliography III

M. Bibinger and M. Trabs.
On central limit theorems for power variations of the solution to the stochastic heat equation.
In A. Steland, E. Rafajłowicz, and O. Okhrin, editors, *Stochastic Models, Statistics and Their Applications*, pages 69–84, Cham, 2019. Springer International Publishing.

M. Bibinger and M. Trabs.
Volatility estimation for stochastic PDEs using high-frequency observations.
Stochastic Processes and their Applications, 130(5):3005–3052, 2020.

I. Cialenco, F. Delgado-Vences, and H.-J. Kim.
Drift estimation for discretely sampled SPDEs.
Stoch PDE: Anal Comp, 8:895–920, 2020.

I. Cialenco and N. Glatt-Holtz.
Parameter estimation for the stochastically perturbed Navier-Stokes equations.
Stochastic Process. Appl., 121(4):701–724, 2011.

Bibliography IV

I. Cialenco and Y. Huang.
A note on parameter estimation for discretely sampled SPDEs.
Stochastics and Dynamics, 20(3):2050016, 2020.

Igor Cialenco, Ruimeng Hu, and Quyuan Lin.
Anisotropic viscosities estimation for the stochastic primitive equations.
September 2023.

C. Chong.
High-frequency analysis of parabolic stochastic PDEs with multiplicative noise: Part I.
Preprint. arXiv:1908.04145, 2019.

C. Chong.
High-frequency analysis of parabolic stochastic PDEs.
Ann. Statist., 48(2):1143–1167, 2020.

Bibliography V

I. Cialenco.
Statistical inference for SPDEs: an overview.
Statistical Inference for Stochastic Processes, 21(2):309–329, 2018.

Igor Cialenco and Hyun-Jung Kim.
Parameter estimation for discretely sampled stochastic heat equation driven by space-only noise.
Stochastic Processes and their Applications, 143:1–30, 2022.

I. Cialenco, H.-J. Kim, and S. V. Lototsky.
Statistical analysis of some evolution equations driven by space-only noise.
Statistical Inference for Stochastic Processes, 23(1):83–103, 2020.

Igor Cialenco, Hyun-Jung Kim, and Gregor Pasemann.
Statistical analysis of discretely sampled semilinear SPDEs: a power variation approach.
Stochastics and Partial Differential Equations: Analysis and Computations, jan 2023.

Bibliography VI

Rama Cont and Marvin S. Müller.
A stochastic partial differential equation model for limit order book dynamics.
SIAM Journal on Financial Mathematics, 12(2):744–787, jan 2021.

R. Cont.
Modeling term structure dynamics: an infinite dimensional approach.
Int. J. Theor. Appl. Finance, 8(3):357–380, 2005.

Rene A. Carmona and Michael R. Tehranchi.
Interest Rate Models: an Infinite Dimensional Stochastic Analysis Perspective.
Springer Berlin Heidelberg, 2006.

Damir Filipović.
Consistency Problems for Heath-Jarrow-Morton Interest Rate Models.
Springer Berlin Heidelberg, 2001.

Bibliography VII

David Heath, Robert Jarrow, and Andrew Morton.
Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation.
Econometrica, 60(1):77, jan 1992.

Ben Hambly and Nikolaos Kolliopoulos.
Stochastic evolution equations for large portfolios of stochastic volatility models.
SIAM Journal on Financial Mathematics, 8(1):962–1014, 2017.

Ben Hambly, Jasdeep Kalsi, and James Newbury.
Limit order books, diffusion approximations and reflected SPDEs: From microscopic to macroscopic models.
Applied Mathematical Finance, 27(1-2):132–170, mar 2020.

Bibliography VIII

M. Huebner, R. Khasminskii, and B. L. Rozovskii.
Two examples of parameter estimation for stochastic partial differential equations.
In *S.Cambanis, J. K. Ghosh, R. L. Karandikar, P. K. Sen eds, Stochastic processes: A Festschrift in Honour of G. Kallianpur*, pages 149–160. Springer, New York, 1993.

M. Huebner and B. L. Rozovskii.
On asymptotic properties of maximum likelihood estimators for parabolic stochastic PDE's.
Probab. Theory Related Fields, 103(2):143–163, 1995.

Ben Hambly and Andreas Søjmark.
An SPDE model for systemic risk with endogenous contagion.
Finance and Stochastics, 23(3):535–594, jun 2019.

Bibliography IX

F. Hildebrandt and M. Trabs.
Parameter estimation for SPDEs based on discrete observations in time and space.
Electron. J. Statist., 15(1):2716–2776, 2021.

Florian Hildebrandt and Mathias Trabs.
Nonparametric calibration for stochastic reaction–diffusion equations based on discrete observations.
Stochastic Processes and their Applications, 162:171–217, 2023.

Sigrid Källblad, Jan Obłój, and Thaleia Zariphopoulou.
Dynamically consistent investment under model uncertainty: the robust forward criteria.
Finance and Stochastics, 22(4):879–918, jul 2018.

Praveen Kolli and Mykhaylo Shkolnikov.
SPDE limit of the global fluctuations in rank-based models.
The Annals of Probability, 46(2), mar 2018.

Bibliography X

Z. M. Khalil and C. Tudor.
Estimation of the drift parameter for the fractional stochastic heat equation via power variation.
Modern Stochastics: Theory and Applications, 6(4):397–417, 2019.

Z. M. Khalil and C. Tudor.
On the distribution and q-variation of the solution to the heat equation with fractional Laplacian.
Probability and Mathematical Statistics, 39(2), 2019.

Y. Kaino and M. Uchida.
Parametric estimation for a parabolic linear SPDE model based on sampled data.
Journal of Statistical Planning and Inference, 211:190–220, March 2021.

Shigeo Kusuoka.
Term structure and SPDE.
In *Advances in Mathematical Economics*, pages 67–85. Springer Japan, 2000.

Bibliography XI

Sergey V Lototsky, Henry Schellhorn, and Ran Zhao.
An infinite-dimensional model of liquidity in financial markets.
Probability, Uncertainty and Quantitative Risk, 6(2):117, 2021.

M. Musiela and T. Zariphopoulou.
Portfolio choice under space-time monotone performance criteria.
SIAM Journal on Financial Mathematics, 1(1):326–365, jan 2010.

Marek Musiela and Thaleia Zariphopoulou.
Stochastic partial differential equations and portfolio choice.
In *Contemporary Quantitative Finance*, pages 195–216. Springer Berlin Heidelberg, 2010.

I. Nourdin and G. Peccati.
Normal approximations with Malliavin calculus, from Stein's method to universality, volume 192 of *Cambridge Tracts in Mathematics*.
Cambridge University Press, 2012.

Bibliography XII

Sergey Nadtochiy and Thaleia Zariphopoulou.
Optimal contract for a fund manager with capital injections and endogenous trading constraints.
SIAM Journal on Financial Mathematics, 10(3):698–722, jan 2019.

L. I. Piterbarg and B. L. Rozovskii.
On asymptotic problems of parameter estimation in stochastic PDE's: discrete time sampling.
Math. Methods Statist., 6(2):200–223, 1997.

G. Pasemann and W. Stannat.
Drift estimation for stochastic reaction-diffusion systems.
Electronic J of Statistics, 14(1):547–579, 2020.

J. Pospíšil and R. Tribe.
Parameter estimates and exact variations for stochastic heat equations driven by space-time white noise.
Stoch. Anal. Appl., 25(3):593–611, 2007.

Bibliography XIII

P. Santa-Clara and D. Sornette.
The dynamics of the forward interest rate curve with stocahstic string shocks.
The review of financial studies, 14(1):149–185, Spring 2001.

R. Shevchenko, M. Slaoui, and C. Tudor.
Generalized k-variations and Hurst parameter estimation for the fractional wave equation via malliavin calculus.
Journal of Statistical Planning and Inference, 207:155–180, 2020.

Breuer-Major Theorem [NP12, Theorem 7.2.4].

Theorem

Let $Y = \{Y_k\}_{k\in\mathbb{Z}}$ *be a centered stationary Gaussian sequence with unit variance, and* $f(x) = \sum_{q=d}^{\infty} a_q H_q(x),\ a_q \in \mathbb{R}$, *where* H_q *is the* q*-th Hermite polynomial. Assume that*

$$\sum_{\ell\in\mathbb{Z}} |\rho(\ell)|^d < \infty, \tag{3.1}$$

where $\rho(\ell) = \mathbb{E}\left(Y_0 Y_\ell\right),\ \ell \in \mathbb{Z}$. *Then,*

$$\text{w}-\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} f(Y_k) = \mathcal{N}\left(0, \sum_{q=d}^{\infty} q! a_q^2 \sum_{\ell\in\mathbb{Z}} \rho(\ell)^q\right).$$

The noise $W(t,x)$

$W(t)$ is a `cylindrical Brownian motion`, if it is an $\mathcal{H} = L^2(\Lambda)$-valued continuous Gaussian process with $W(0)=0$, and *covariance structure*

$$\mathbb{E}[\langle W(t), g\rangle_\Lambda \cdot \langle W(s), f\rangle_\Lambda] = \min(t,s) \cdot \langle f, g\rangle_\Lambda$$

$\dot{W}(t,x),\ t \geq 0, x \in \Lambda \subset \mathbb{R}^d$ is called `space-time white noise`:

- ◇ a zero mean Gaussian process with covariance

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y)$$

- ◇ $\dot{W}(t,x) = \sum_{k\geq 1} h_k(x)\dot{w}_k(t)$,
  where $\{h_k(x)\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\Lambda)$, and $\{\dot{w}_k\}_{k\in\mathbb{N}}$ are independent 1d white-noises.

- ◇ a random generalized function on $L_2((0,T)\times\Lambda)$
  $\dot{W}(t,x) = \int_0^T \int_\Lambda f(t,x)\,\mathrm{d}W(t,x)$.