Why is Cash U-Shaped in Firm Size?

Ali Kakhbod	Max Reppen	Tarik Umar	Hao Xing
UC Berkeley	Boston University	Rice University	Boston University

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Motivation

Amadeus data:

- 11.5 million private firms, 45 European countries,
- $100 trillion assets
- 42.8% aggregated corporate assets, 61.8% total workforce in Europe
- 2011 - 2022
- Average Amadeus firms are much smaller than U.S. public firms
  mean Amadeus firm has $1.9-million in assets
  mean U.S public firm in COMPUSTAT has $849-million in assets

Cash holdings are U-shaped in firm size

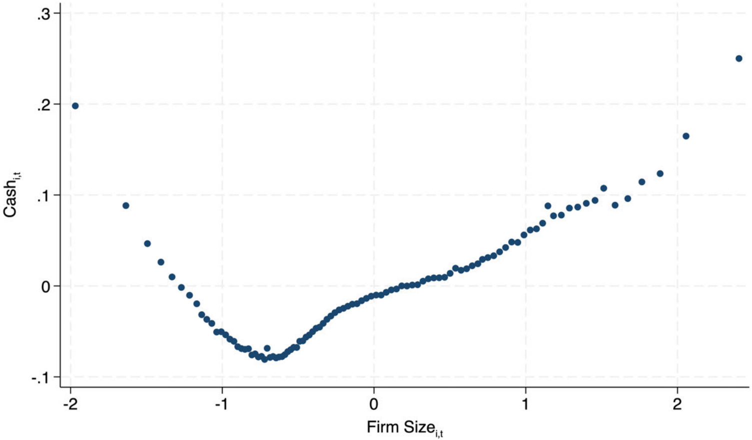
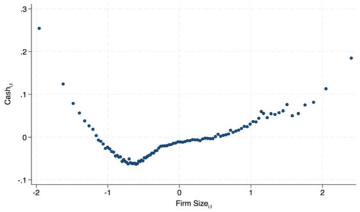
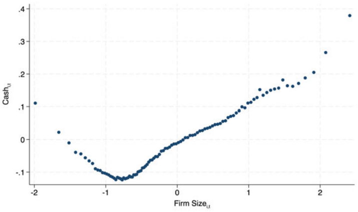


Figure: **Cash holdings and firm size**

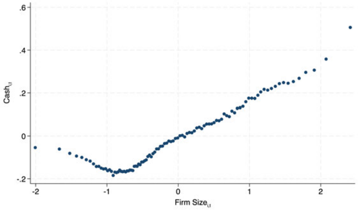
Firms in different size categories



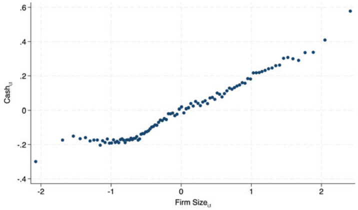
(a) SMALL firms



(b) MEDIUM SIZED firms



(c) LARGE firms



(d) VERY LARGE firms

Equity issuance

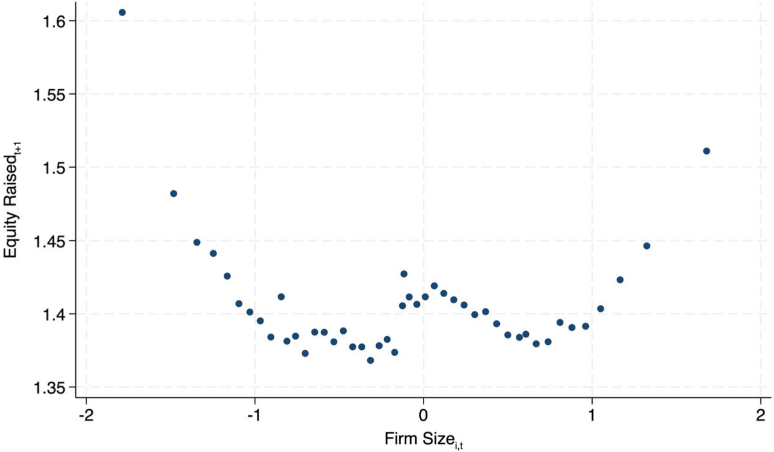


Figure: **Equity raises and firm size**

What we do

Dynamic firm model

- Investment, payout, default, equity issuance, cash holdings

Model is not homothetic in firm size

- Decreasing return to scale
- Fixed equity issuance cost, independent of firm size

Model is calibrated to empirical moments using SMM

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What we obtain:

Firm's demand for cash is U-shaped in firm size

- Small firm has strong motive to invest, but face low cash flow and high issuance cost
- Large firm has strong motive to hedge cash flow risk

Model generated U-shape is close to the empirical counterpart

Literature

- Jeanblanc-Picqué and Shiryaev (1995)
- Radner and Shepp (1996)
- Asmussen and Taksar (1997)

- Décamps, Mariotti, Rochet, Villeneuve (2011)
- Bolton, Chen, Wang (2011)
- Anderson, Charverhill (2012)

- Jiang and Pistorius (2012)
- Akyildirim, Gü, Rochet, and Soner (2014)
- Reppen, Rochet, and Soner (2020)

Cash Flows

$$dZ_t = \mu dt + \sigma dW_t$$

$$dY_t = k_t^{\alpha} dZ_t$$

- dZ_t is the cash flow shock
  - W is one-dimensional Brownian motion
  - μ is the drift
  - σ is the volatility

- dY_t is the firm's cash flows
  - k is capital
  - $\alpha \in (0, 1)$ is a scaling parameter. Production exhibits diminishing returns to scale

Dynamics of Capital Stock

$$dk_t = (i_t - \delta k_t)dt + \sigma_K k_t dB_t$$

$$g(k, i) = \frac{\theta}{2}\left(\frac{i}{k}\right)^2 k$$

- dk_t is the net investment in the capital stock
  - i_t is amount invested
  - δ is the depreciation rate (%)
  - B is one-dimensional Brownian motion independent of W
- $g(k, i)$ is the standard quadratic adjustment costs
  - θ measures the degree of adjustment cost

Equity Issuance

$$\lambda(I) = \lambda_f + \lambda_p I$$

- I is the amount issued
- λ_p is the proportional component of issuance costs
- λ_f is the constant component of issuance costs
  - A fixed component makes issuance relatively more costly for small firms

Cash Reserve Dynamics

$$dc_t = (r - \lambda_c)c_t dt + dY_t - i_t dt - g(k_t, i_t)dt - dD_t + dI_t.$$

- $(r - \lambda_c)$ is the return on cash less a liquidity premium
- dY_t is cash flows generated
- $-i_t d_t$ is investment
- $-g(k_t, i_t)$ is adjustment cost
- $-dD_t$ is the cumulative payout
- dI_t is cumulative issuance

Default Time

$$\tau = \inf\{t \geqslant 0 : c_t < 0\}$$

- Default value at τ: ℓk_τ
- Capital stock fire sold k_τ
- Recover rate ℓ

The Firm's Problem

$$\sup_{i\geqslant 0, D, \{\sigma_j, I_j\}} E\Big[\int_0^{\tau} e^{-rs} dD_s - \sum_j e^{-r\sigma_j} \big(I_j + \lambda(I_j) \big) + 1_{\{\tau<\infty\}} e^{-r\tau} \ell k_\tau \Big],$$

- Equityholders choose investment, dividends, and equity issuance
- Impulse control problem with two state variables: capital size k and cash reserve c

HJB Equation

$$0 = \min\Big\{ \underbrace{\partial_c V - 1}_{\text{dividend payout}},\ \underbrace{V(k,c) - \sup_{I \geqslant 0}\big[V(k,c+I) - I - \lambda(I)\big]}_{\text{equity issuance}},$$

$$\underbrace{rV - \sup_{i \geqslant 0}\Big\{\big[i - \delta k\big]\partial_k V + \big[(r-\lambda_c)c + k^\alpha \mu - i - g(k,i)\big]\partial_c V + \frac{1}{2}k^{2\alpha}\sigma^2 \partial^2_{cc} V + \frac{1}{2}k^2\sigma_k^2 \partial^2_{kk} V\Big\}}_{\text{continuation}} \Big\}$$

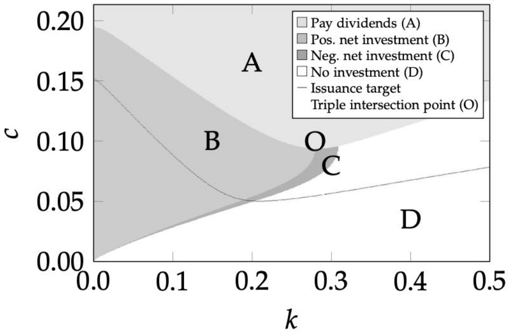
At $c = 0$:

$$0 = \min\Big\{ \underbrace{V(k,0) - \ell k}_{\text{liquidation}},\ \underbrace{V(k,0) - \sup_{I \geqslant 0}\big[V(k,I) - I - \lambda(I)\big]}_{\text{equity issuance}} \Big\}$$

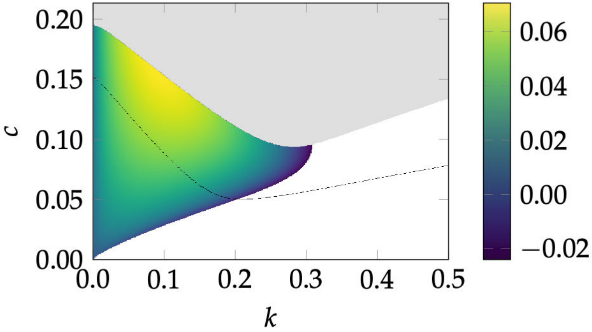
Solution

- Numerically solved through policy iteration
  - Policy evaluation
  - Policy update
- We prove the uniqueness of the model solution and convergence of the numeric algorithm to the value function
  - Provide a comparison theorem for convergence

Firm's stragegies

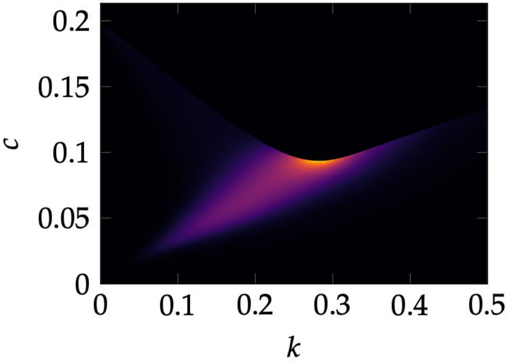


(a) Strategies in different regions

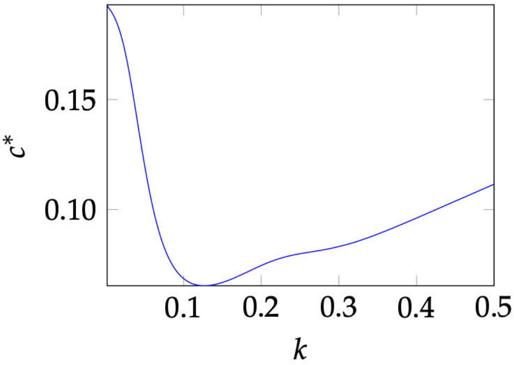


(b) Net investment heat map

Firm's cash policy and firm size

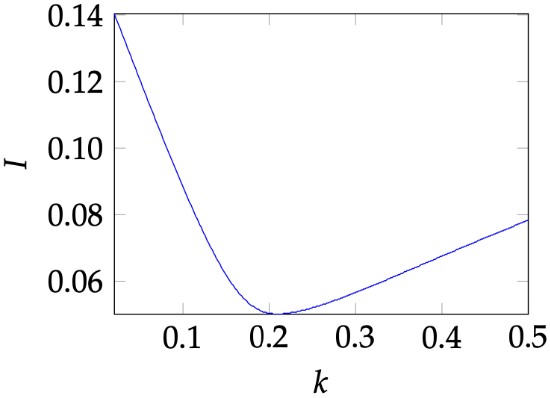


(a) The density in the capital-cash (k, c) space



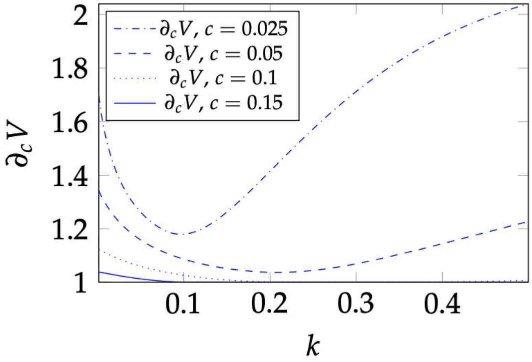
(b) Cash holdings are U-shaped in firm size

Firm's issuance policy and firm size

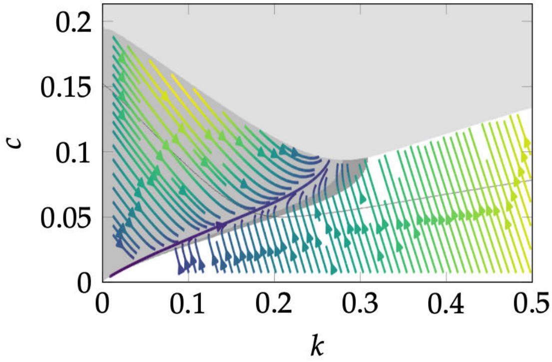


(c) Issuance amounts are U-shaped in firm size (raising cash)

Marginal value of cash

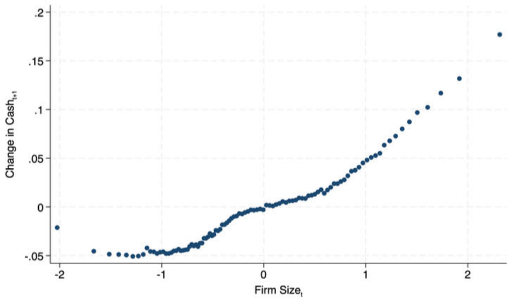


(a) Marginal value of cash

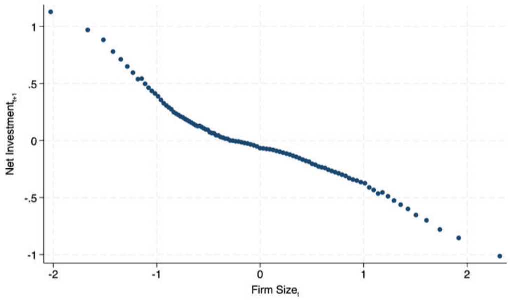


(b) State dynamics drift

Empirical evidence



(a) Growth in Cash (Amadeus)



(b) Net Investment (Amadeus)

Larger firm has stronger precautionary motive

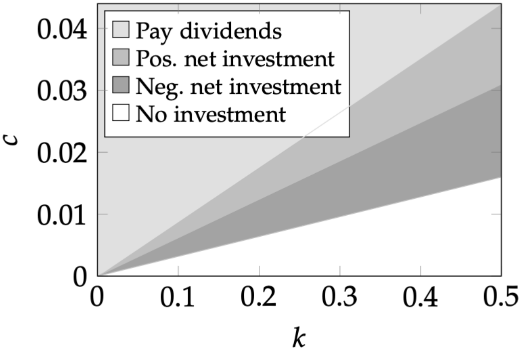
$$\text{Cash}_{i,t} = \beta_1 \text{VIX}_{i,t} + \beta_2 \text{Firm Size}_{i,t-1} + \beta_3 \text{VIX}_{i,t} \times \text{Firm Size}_{i,t-1} + \epsilon_{i,t}.$$

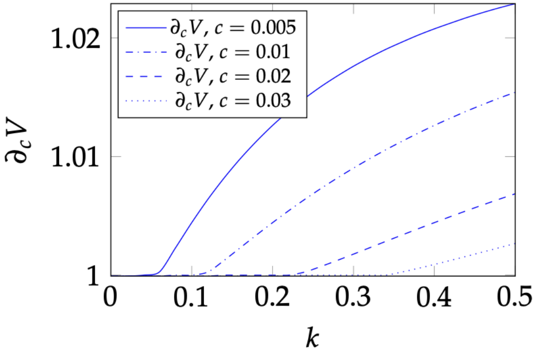
	$\text{Cash}_{i,t}$ (1)	$\Delta\text{Cash}_{i,t}$ (2)
VIX_t	0.025***	
	(0.006)	
$\text{VIX}_t \times \text{Firm Size}_{i,t-1}$	0.004**	
	(0.001)	
ΔVIX_t		0.002*
		(0.001)
$\Delta\text{VIX}_t \times \text{Firm Size}_{i,t-1}$		0.002***
		(0.000)
$\text{Firm Size}_{i,t-1}$	0.011	0.048***
	(0.031)	(0.005)
% Adjusted R^2	2.9	0.4
Observations	66,181,226	65,086,184

Homothetic models

- Constant return to scale $(\alpha = 1)$
- fixed issuance cost proportional to size $(\lambda_f(k) \propto k)$

$$V(c,k) = kv(c/k) \quad \text{and} \quad \partial_c V(c,k) = v'(c/k)$$





Simulated method of moments (SMM)

We use SMM to calibrate

- A: scaling parameter for cash flow μ and σ
- θ: investment adjustment cost parameter
- α: decreasing return to scale parameter
- λ_f: fixed issuance cost

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For each parameter set $\Phi = (A, \theta, \alpha, \lambda_f)$

- solve firm's optimal policies
- simulate firm dynamics starting from an initial distribution (k_0, c_0) for 10 years
- treat simulations starting from the same point as trajectory of the same firm
- calculate 4 firm-level moments, denote cross-sectional average as $\Psi(\Phi)$

SMM cont.

4 firm-level moments:

- average cash / capital
- standard deviation of cash / capital
- percent change in capital
- average cash-flow / capital.

Denote $\{X_i\}_{i=1,\dots,N}$ the set of firm-level data.

$$\Psi_D = \frac{1}{N}\sum_{i=1}^{N} X_i$$

SMM cont.

4 firm-level moments:

- average cash / capital
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Denote $\{X_i\}_{i=1,\dots,N}$ the set of firm-level data.

$$\Psi_D = \frac{1}{N}\sum_{i=1}^{N} X_i$$

First step optimization:

$$\tilde{\Phi} = \arg\min_{\Phi}(\Psi(\Phi) - \Psi_D)'(\Psi(\Phi) - \Psi_D)$$

SMM cont.

Update weight matrix $\widehat{W}$

$$\widehat{W}^{-1} = \Omega_D + (\Psi(\tilde{\Phi}) - \Psi_D)(\Psi(\tilde{\Phi}) - \Psi_D)',$$

where Ω_D is the covariance matrix of $\{X_i\}_{i=1,\dots,N}$

SMM cont.

Update weight matrix $\widehat{W}$

$$\widehat{W}^{-1} = \Omega_D + (\Psi(\tilde{\Phi}) - \Psi_D)(\Psi(\tilde{\Phi}) - \Psi_D)',$$

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Second step optimization:

$$\hat{\Phi} = \arg\min_{\Phi}(\Psi(\Phi) - \Psi_D)\widehat{W}(\Psi(\Phi) - \Psi_D)'$$

SMM cont.

Update weight matrix $\widehat{W}$

$$\widehat{W}^{-1} = \Omega_D + (\Psi(\tilde{\Phi}) - \Psi_D)(\Psi(\tilde{\Phi}) - \Psi_D)',$$

where Ω_D is the covariance matrix of $\{X_i\}_{i=1,\dots,N}$

Second step optimization:

$$\hat{\Phi} = \arg\min_{\Phi}(\Psi(\Phi) - \Psi_D)\widehat{W}(\Psi(\Phi) - \Psi_D)'$$

The asymptotic distribution of $\hat{\Phi}$ is given by

$$\sqrt{N}(\hat{\Phi} - \Phi_0) \sim N(0, \Omega),$$

where Ω is determined by gradient of $\Psi(\Phi) - \Psi_D$ and $\widehat{W}$

Calibrated model

Panel A: Calibrated Parameters

Investment adjustment cost (θ)	0.100	(0.0002)
Diminishing returns to scale (α)	0.850	(0.0003)
Fixed component of issuance costs (λ_f)	0.072	(0.0002)
Scale parameter for the cash flow parameters μ and σ (A)	0.750	(0.0002)

Panel B: In Sample Moments

	Sample	Model
Avg. firm-level mean $\text{Cash}_t/(\text{Total Assets}_t\text{-Cash}_t)$ (%)	51.0	47.0
Avg. firm-level standard deviation of $\text{Cash}_t/(\text{Total Assets}_t\text{-Cash}_t)$ (%)	30.5	34.3
Avg. firm-level mean percentage change in $\text{Total Assets}_t\text{-Cash}_t$ (%)	14.2	12.4
Avg. firm-level mean $\text{Cash Flow}_t/(\text{Total Assets}_{t-1}\text{-Cash}_{t-1})$ (%)	18.7	17.6

Panel C: Out of Sample Moment

	Sample	Model
$\beta(\text{Cash}, \text{Capital}^2)$	0.034	0.028

Conclusion

Dynamic firm model that is not homothetic in size: two-state model

- Decreasing return to scale + costly equity financing
- strong investment motive among small firms, strong heging motive among big firms
- firm cash holding is U-shaped in firm size
- firm equity issuance size is U-shaped in firm size


Happy
THANKSGIVING!