Unwinding Stochastic Order Flow

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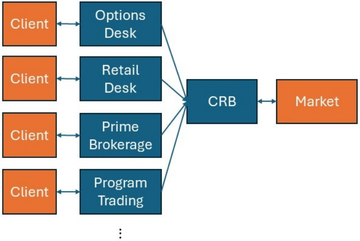
Motivation: Central Risk Book (CRB)

- Centralized trading unit recently established in many large banks, trading firms



CRB aggregates order flow from other business units within the organization (in-flow)

- In-flow is a stochastic process. Uncontrolled.

- CRB nets opposite orders (internalization)
- Unwinds outstanding positions in the market (out-flow)

→ Optimal execution problem for a stochastic position

- Another use case: market maker aggressively unwinding inventory

Some Stylized Facts to Capture

- FX dealers achieve high double-digit internalization rates
  Bank of England '14
- For institutional size orders in large cap stocks,
  $\approx$ 30 bps of price impact costs
  $\approx$ 5 bps of instantaneous costs

  Nasdaq Guide for Trading Interns '22
- Intraday impact decay: half of a trade's impact dissipates over 1-2 h
  Horst et al. '19, Muhle-Karbe et al. '22
- Price impact parameters vary intraday
  Cont et al. '13, Fruth et al. '13
- Volume in closing auction is a double-digit percentage of total
  Bouchaud et al. '18

Deliverables

- Flexible model that can be calibrated to a particular flow
- Principled, implementable unwind strategy
  Guarantee for no-price-manipulation
  Formula for expected cost
- Sensitivity to in-flow characteristics
  Misspecification analysis

Outline

1. Model and Synopsis of Results
2. Solution
3. Numerical Simulations

Model

- Finite horizon $T > 0$ (one trading day)
- Cumulative in-flow orders: process Z,

$$dZ_t = -\theta Z_t \, dt + \sigma \, dW_t, \quad Z_0 = z$$

- Can be mean-reverting or trending. Historical internal data
- Model is tractable for general martingale driver

- Cumulative out-flow process Q is our control
- Choose Q such as to minimize expected transaction cost
- Liquidation constraint $Q_T = Z_T$

Transaction Costs

Two reasons to hold inventory:

- Increase chance of netting, reduce spread cost
- Unwind slowly to reduce impact

Transaction cost modeling:

- S_t "unaffected" market price
- **Assumption:** S is a martingale
- $P_t = P_t(Q, S)$ actual execution price
- Execution cost

$$\int_0^T P_t \, dQ_t$$

Instantaneous cost

- Execution price $P_t = S_t + \frac{1}{2}\varepsilon \dot{Q}_t$ with $\varepsilon > 0$
- Execution cost

$$\int_0^T P_t \, dQ_t = \int_0^T S_t \, dQ_t + \frac{1}{2}\varepsilon \int_0^T \dot{Q}_t^2 \, dt.$$

- Imposes absolutely continuous trading

  Bertsimas & Lo '98, Almgren & Chriss '01, ...

Persistent (decaying) impact

- Impact process $dY_t = -\beta Y_t \, dt + \lambda \, dQ_t$
- Execution price $P_t = S_t + \frac{1}{2}(Y_{t-} + Y_t)$
- Execution cost

$$\int_0^T P_t \, dQ_t = \int_0^T S_t \, dQ_t + \int_0^T Y_{t-} \, dQ_t + \frac{1}{2}[Y, Q]_T.$$

Obizhaeva & Wang '13, ...

Transaction Cost Model

- Regular LOB trading for $t \in (0, T)$
- Both instantaneous cost and persistent impact cost (permanent cost)
  Graewe & Horst '17
- Absolutely continuous trading during the day, $t \in (0, T)$

- Opening auction (cross) at $t = 0$
- Closing auction (cross) at $t = T$
- No instantaneous cost (no spread), only impact cost

$\rightarrow$ Block trades in the auctions

- Generic trading strategy Q is a triplet $(J_0, (q_t), J_T)$:

$$J_0 = \Delta Q_0, \qquad q_t = \dot{Q}_t, \quad t \in (0, T), \qquad J_T = \Delta Q_T$$

- Impact process

$$dY_t = -\beta Y_t\, dt + \lambda\, dQ_t, \quad Y_{0-} = y$$

- Execution price

$$P_t := \begin{cases} S_t + \frac{1}{2}(Y_{t-} + Y_t), & t \in \{0, T\} \\ S_t + Y_t + \frac{1}{2}\varepsilon q_t, & t \in (0, T). \end{cases}$$

- Expected execution cost

$$\mathcal{C}(J_0, q, J_T) = \mathbb{E}\left[\int_0^T P_t\, dQ_t\right] = \mathbb{E}\left[P_0 J_0 + \int_0^T P_t q_t\, dt + P_T J_T\right]$$

- **Problem:** minimize $\mathcal{C}(J_0, q, J_T)$ subject to $Q_T = Z_T$

Synopsis

- Semi-explicit optimal strategy
- Intra-day trading speed has feedback-form

$$\boxed{q_t = f_t X_t + g_t Y_t + h_t Z_t}$$

$$X_t \dots \text{ inventory}, \quad Y_t \dots \text{ impact state}, \quad Z_t \dots \text{ cum. in-flow}$$

- Time-dependent coefficients f_t, g_t, h_t determined by ODE (closed form for constant liquidity parameters)
- h_t is the adjustment to stochastic in-flow

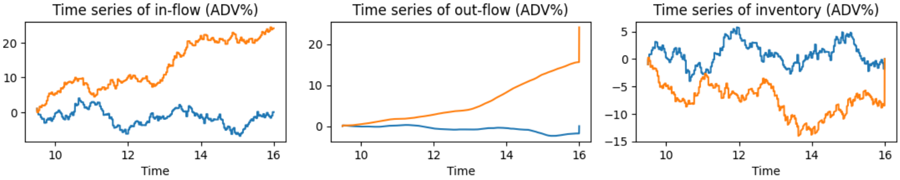
Synopsis

Having a computable model allows us to study (model-dependent) trading metrics from an input–output perspective

Core trading metrics strongly depend on in-flow characteristics:

- Momentum requires more aggressive trading, increasing costs. Reversion leads to more internalization and more warehousing
- Misspecification: overestimating momentum sharply increases costs
- Expected trading costs (per order notional) are minimized at a particular in-flow volatility

Example Paths



Two realizations of the same model ($\theta = 0$) illustrating extreme regimes.

Outline

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Solution Strategy

- Fix arbitrary J_0 and q: outstanding position before closing auction has to be the final block trade: $J_T = Q_{T-} - Z_{T-}$
- Replace liquidation constraint by cost of that block trade

- After reformulation, that cost is a function of the terminal states: standard LQ control problem

$\rightarrow$ Solution q^* and cost $\mathcal{C}(J_0, q^*, J_T^*)$

- Finally, optimize that cost $\mathcal{C}(J_0, q, J_T)$ over J_0

Rewriting the Execution Cost

Let $(J_0, q, J_T) \in \mathcal{A}$. Define state processes $(X_t, Y_t^c, Z_t)_{t\in[0,T]}$ by

$$\begin{cases} dX_t = q_t\,dt - dZ_t, & X_0 = J_0 - z \quad \text{outstanding position} \\ dY_t^c = (-\beta Y_t^c + \lambda q_t)\,dt, & Y_0^c = y + \lambda J_0 \quad \text{impact state} \\ dZ_t = -\theta Z_t\,dt + \sigma_t\,dW_t, & Z_0 = z \quad \text{cumulative in-flow} \end{cases}$$

The expected execution cost is

$$\mathcal{C}(J_0, q, J_T) = \frac{1}{2}\mathbb{E}\bigg\{\int_0^T \left[\frac{2\beta}{\lambda}(Y_t^c)^2 + \varepsilon q_t^2\right] dt + \frac{1}{\lambda}(Y_T^c - \lambda X_T)^2 - \frac{y^2}{\lambda} + S_T Z_T\bigg\}$$

Dynamic Programming

- Start at time t, initial states (x, y, x):

$$\begin{cases} dX_s = q_s\, ds - dZ_s, & X_t = x \quad \text{outstanding position} \\ dY_s^c = (-\beta Y_s^c + \lambda q_s)\, ds, & Y_t^c = y \quad \text{impact state} \\ dZ_s = -\theta Z_s\, ds + \sigma\, dW_s, & Z_t = z \quad \text{cumulative in-flow} \end{cases}$$

- Auxiliary LQ problem

$$v(t,x,y,z) = \inf_q \frac{1}{2}\mathbb{E}\left\{\int_t^T \left(\frac{\beta}{\lambda}(Y_s^c)^2 + \frac{1}{2}\varepsilon q_s^2\right) ds + \frac{1}{2\lambda}(Y_T^c - \lambda X_T)^2\right\}$$

- Quadratic value function v
- Linear optimal feedback control: $q^*(t,x,y,z) = f_t x + g_t y + h_t z$

Time-varying Market Parameters

- Liquidity increases over the course of the trading day (regular hours)

$\rightarrow$ Time-dependent λ_t

- Auctions empirically more liquid than adjacent regular trading

- Also make ε_t, β_t time-dependent (bounded)
- λ_t, ε_t bounded away from zero

Fast increase in liquidity can give rise to arbitrage ("price manipulation")

Huberman & Stanzl '04

- E.g., suppose $\beta, \varepsilon \equiv 0$ and $\lambda_{t_0} > \lambda_{t_1}$ for some $t_0 < t_1$

$\rightarrow$ Buy at t_0 and sell at t_1. Or: Sell at t_0 and buy at t_1.

- In general, trade-off between β and $\dot{\lambda}$

Parameter Restrictions

- λ differentiable (regular hours)
- λ_{0-} for opening auction

- **Assumption:**

$$2\beta_t + \dot{\gamma}_t > 0 \quad (\dot{\gamma}_t := \dot{\lambda}_t / \lambda_t) \quad \text{and} \quad \lambda_{0-} \leq \lambda_0$$

Fruth, Schöneborn & Urusov '14

- We use λ_T for the closing auction (liquidity jump causes arbitrage)

Execution Cost

For any $(J_0, q, J_T) \in \mathcal{A}$, the expected execution cost is

$$\mathcal{C}(J_0, q, J_T) = \frac{1}{2}\mathbb{E}\Bigg\{ \int_0^T \left[\frac{2\beta_t + \dot{\gamma}_t}{\lambda_t}(Y_t^c)^2 + \varepsilon_t q_t^2\right] dt + \frac{1}{\lambda_T}(Y_T^c - \lambda_T X_T)^2 + (Y_0^c)^2\left[\frac{1}{\lambda_{0-}} - \frac{1}{\lambda_0}\right] - \frac{y^2}{\lambda_{0-}} + S_T Z_T \Bigg\},$$

where $(X_t, Y_t^c, Z_t)_{t\in[0,T]}$ are defined by

$$\begin{cases} dX_t = q_t\, dt - dZ_t, & X_0 = J_0 - z \\ dY_t^c = (-\beta_t Y_t^c + \lambda_t q_t)\, dt, & Y_0^c = y + \lambda_{0-} J_0 \\ dZ_t = -\theta_t Z_t\, dt + \sigma_t\, dW_t, & Z_0 = z \end{cases}$$

- $\lambda_{0-} \leq \lambda_0$ and $2\beta_t + \dot{\gamma}_t > 0$ make problem (strictly) convex

(No) Price Manipulation

There is no “price manipulation” strategy:

Corollary: Let $Z \equiv 0$ and $y = 0$. Then $Q \equiv 0$ is the only admissible strategy with zero cost.

Remark:

- Starting with non-zero inventory, round trips could nevertheless be profitable: “transaction-triggered price manipulation”
  Alfonsi, Schied & Slynko ’11, Gatheral, Schied & Slynko ’12
- Starting with non-zero impact state, it will often be optimal to trade, even if $Z \equiv 0$

Solution on $(0, T)$ for fixed J_0

Fix $t \in [0, T)$ and consider $(X_s, Y_s^c, Z_s)_{s\in[t,T]}$ started at (x, y, z). Then

$$v(t, x, y, z) := \inf_q \frac{1}{2}\mathbb{E}\left\{ \int_t^T \left[\frac{2\beta_s + \dot{\gamma}_s}{\lambda_s}(Y_s^c)^2 + \varepsilon_s q_s^2\right] ds \right. \\ \left. + \frac{1}{\lambda_T}(Y_T^c - \lambda_T X_T)^2 \right\}$$

is of the form

$$v(t, x, y, z) = \frac{1}{2}A_t x^2 + B_t xy + \frac{1}{2}C_t y^2 + D_t xz + E_t yz + \frac{1}{2}F_t z^2 + K_t$$

where $A_t, B_t, \ldots, K_t$ are defined below.

Riccati ODE System

Proposition: The Riccati ODE system

$$
\begin{cases}
\dot{A}_t = \varepsilon_t^{-1}(A_t + \lambda_t B_t)^2, & A_T = \lambda_T \\
\dot{B}_t = \varepsilon_t^{-1}(A_t + \lambda_t B_t)(B_t + \lambda_t C_t) + \beta_t B_t, & B_T = -1 \\
\dot{C}_t = \varepsilon_t^{-1}(B_t + \lambda_t C_t)^2 + 2\beta_t C_t - \lambda_t^{-1}(2\beta_t + \dot{\gamma}_t), & C_T = \lambda_T^{-1} \\
\\
\dot{D}_t = \varepsilon_t^{-1}(A_t + \lambda_t B_t)(D_t + \lambda_t E_t) - \theta_t(A_t - D_t), & D_T = 0 \\
\dot{E}_t = \varepsilon_t^{-1}(B_t + \lambda_t C_t)(D_t + \lambda_t E_t) - \theta_t(B_t - E_t) + \beta_t E_t, & E_T = 0 \\
\dot{F}_t = \varepsilon_t^{-1}(D_t + \lambda_t E_t)^2 - 2\theta_t(D_t - F_t), & F_T = 0 \\
\\
\dot{K}_t = -\sigma_t^2(A_t - 2D_t + F_t)/2, & K_T = 0
\end{cases}
$$

has a unique solution $(A_t, B_t, C_t, E_t, F_t, K_t)_{t\in[0,T]}$.

Wonham '68, Graewe & Horst '17

Optimal Strategy

The unique optimal control (in feedback form) is

$$\boxed{q^*(t,x,y,z) = f_t x + g_t y + h_t z}$$

where

$$f_t := -\varepsilon_t^{-1}(A_t + \lambda_t B_t)$$
$$g_t := -\varepsilon_t^{-1}(B_t + \lambda_t C_t)$$
$$h_t := -\varepsilon_t^{-1}(D_t + \lambda_t E_t)$$

- f, g only depend on market parameters, not on in-flow characteristics
- h also depends on mean reversion θ, but not on σ

$\Rightarrow$ h is the adjustment for random in-flows

- σ only affects the cost

- Closed-form solution for time-independent liquidity parameters

Properties of the Optimal Strategy

$$\boxed{q^*(t,x,y,z) = f_t x + g_t y + h_t z}$$

- $h = 0$ if $\theta \equiv 0$ (martingale in-flow)

$\rightarrow$ h is the adjustment to in-flows with reversion/momentum

$\rightarrow$ no adjustment for "truthtelling" flow

Under the additional condition $\beta + \dot{\gamma} > 0$ (vs. $2\beta + \dot{\gamma} > 0$):

- h is monotone decreasing wrt. in θ
- $h \geq 0$ if $\theta \leq 0$ (momentum): overtrading
- $h \leq 0$ if $\theta \geq 0$ (mean-reversion)

- $g \leq 0$: high impact state incentivizes selling and vice versa

Properties of the Optimal Strategy (2)

$$\boxed{q^*(t, x, y, z) = f_t x + g_t y + h_t z}$$

Again, under $\beta + \dot{\gamma} > 0$:

- $f \leq 0$: high inventory incentivizes selling and vice versa

Whereas:

- If $\beta + \dot{\gamma} < 0 < 2\beta + \dot{\gamma}$, then $f_t > 0$ for $T - t$ small
- q^* trades in the "wrong" direction even if $y = 0$ and no future in-flow
  "transaction-triggered price manipulation"
  Fruth, Schöneborn & Urusov '14

- In practice, enforce $\beta + \dot{\gamma} > 0$ when using estimated parameters

Optimal Initial Block Trade

Proposition: The optimal block trade J_0 for the opening auction is

$$J_0 = -\frac{g_{0-} + \eta_{0-}}{f_{0-} + \lambda_{0-}\left(g_{0-} + \eta_{0-}\right)}\, y + \frac{f_{0-} - h_{0-}}{f_{0-} + \lambda_{0-}\left(g_{0-} + \eta_{0-}\right)}\, z,$$

where $\eta_{0-} := -\varepsilon_0^{-1}(1 - \lambda_{0-}/\lambda_0) \leq 0$,

$$g_{0-} + \eta_{0-} < 0 \quad \text{and} \quad f_{0-} + \lambda_{0-}\left(g_{0-} + \eta_{0-}\right) < 0.$$

If $\beta_t + \dot{\gamma}_t > 0$ for $t \in [0, T]$, then $f_{0-} - h_{0-} < 0$.

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Trading Metrics

- Key metrics such as impact cost are model-dependent
- Having a reduced-form model allows us to study trading metrics from an input–output perspective
- Practitioners express many quantities *per order notional*

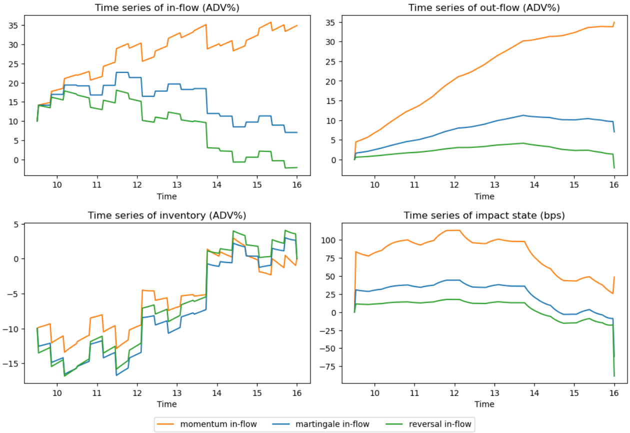
$\Rightarrow$ Our experiments use a finite-variation in-flow

- Internalization is a key metric:

$$\text{internalization rate} = 1 - \frac{\text{total variation of out-flow}}{\text{total variation of in-flow}}$$
$$\approx \text{fraction of in-flow orders netted}$$

- model-independent and client-centric; often used as a proxy for cost

In-Flow Autocorrelation: Sample Paths



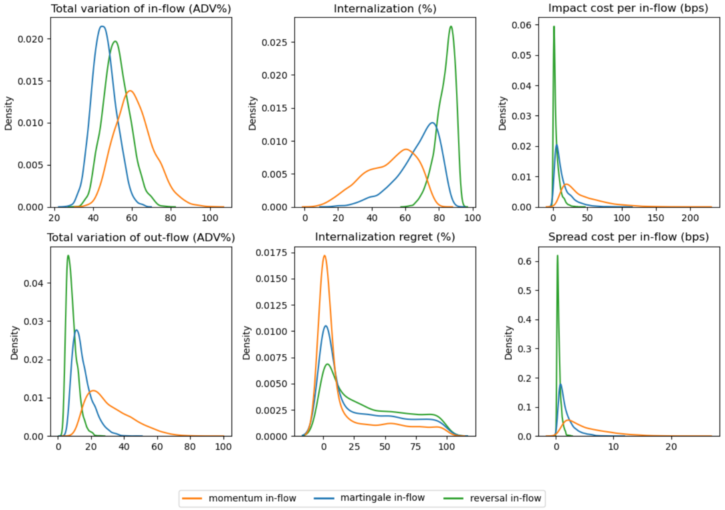
Sensitivity to flow autocorrelation ($\theta = -1, 0, 1$) for an initial inventory (z) and daily flow volatility (σ) of 10% ADV: sample path.

In-Flow Autocorrelation: Averages

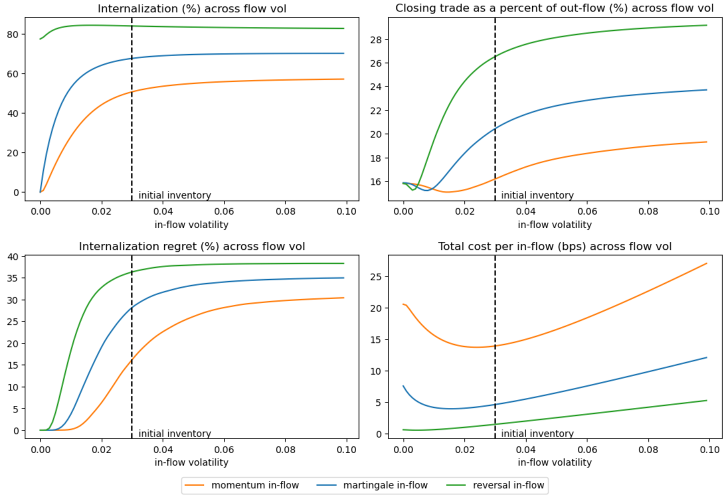
Parameter scans	θ	In-flow (ADV%)	Out-flow (ADV%)	Spread Cost (bps)	Impact cost (bps)	Closing trade (% total)	Internalization (%)
momentum	-1	61	31	4.9	42.6	17	51
martingale	0	46	15	1.7	14.5	21	68
reversal	1	52	9	0.5	4.8	27	84

- Higher autocorrelation leads to more aggressive unwinding
- Internalization drops from 84% to 51%: the strategy trades 3x more volume on the market
- Impact costs increase from 5bps to 43bps: the strategy pays 8x more trading costs
- The closing jump trade shrinks from 27% to 17% of the outflow: the strategy doesn't warehouse as much into the close

In-Flow Autocorrelation: Distributions



In-Flow Volatility



Average metrics' sensitivity to in-volatility (σ) for an initial inventory (z) of 3% ADV and different autocorrelation values (θ).

In-Flow Volatility

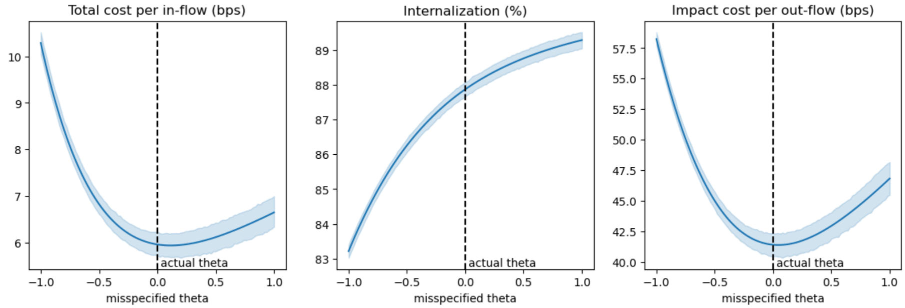
- *some* flow volatility reduces cost per order notional, by promoting internalization
- costs increase with *too much* flow volatility

θ	-1	-0.5	0	0.5	1
optimal σ/z (%)	99	79	55	35	17
Internalization (%)	55	65	72	77	82

Optimal in-flow volatility, as a percentage of inventory, across various θ

Misspecification

The P&L depends on θ, σ, but the strategy only depends on θ. Therefore, there are no misspecification costs for σ.



Actual $\theta = 0$: true in-flow is martingale

Overly aggressive trading sharply increases transaction cost and misses netting opportunities. Preferable to underestimate momentum, causing less aggressive trading.

Thank you