Multivariate Portfolio Choice via Quantiles

Carole Bernard

joint work with
Andrea Perchiazzo and Steven Vanduffel

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Outline of the Talk

1. Optimal Financial Decision Making
   - Role of cost-efficiency
   - Quantile Approach
   - Towards a generalisation to the multivariate case...

2. Optimal Multivariate Financial Decision Making
   - “Multivariate” cost-efficiency - Characterization of optimum
   - Reduction to a one-dimensional problem
   - Numerical approximation

3. Multivariate Risk Sharing via Quantile Approach
   - Theoretical elements
   - Example with a bivariate expected utility
   - Example with a multivariate Yaari investor
Cost-efficiency

- A portfolio/cash-flow/consumption with final payoff $X_T$ (consumption only at time $T$).
- A complete market
- Initial cost of $X_T$ is given by $x_0 = c(X_T) = \mathbb{E}[\xi T X_T]$.

A strategy $X^*_T$ (or a payoff) with cdf $F$ is cost-efficient if any other strategy that generates the same distribution $F$ at the time horizon $T$ costs at least as much, i.e., if it solves

$$\min \{X_T \mid X_T \sim F\} \mathbb{E}[\xi T X_T]$$
Explicit Representation of Cost-efficient Payoffs

**Theorem**

Consider the cost-efficiency problem:

\[
\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]
\]

Assume \(\xi_T\) is continuously distributed, then the optimal strategy is

\[
X_T^\star = F^{-1} (1 - F_{\xi}(\xi_T)).
\]

Note that \(X_T^\star \sim F\) and \(X_T^\star\) is a.s. unique solution.

Intuition of the proof: \[
\frac{\mathbb{E}[\xi_T X_T] - \mathbb{E}[\xi_T] \mu_F}{\text{std}(\xi_T) \sigma_F} = \text{corr}(\xi_T, X_T)
\]
Cost-efficiency & Portfolio Choice (General preferences)

\( V(\cdot) \) denotes the objective function of the agent to maximize (Expected utility, Value-at-Risk, Cumulative Prospect Theory...).

\[
\max_{X_T \mid \mathbb{E}[\xi_X X_T] = \omega_0} V(X_T).
\] (1)

Preferences \( V(\cdot) \) are assumed to be

- **non-decreasing**: \( X_T \geq Y_T \ \text{a.s.} \Rightarrow V(X_T) \geq V(Y_T) \)
- **law-invariant**: \( X_T =_d Y_T \Rightarrow V(X_T) = V(Y_T) \)

Equivalently, \( V(\cdot) \) respects **First-order stochastic dominance**

**Theorem: Optimal strategies are cost-efficient**

If an optimum \( X_T^* \) of (1) exists, let \( F \) be its cdf. Then, \( X_T^* \) is the cheapest way (cost-efficient) to achieve \( F \) at \( T \), i.e.

\[
X_T^* = F^{-1}(1 - F_\xi(\xi_T)) \text{ where } F_\xi \text{ is the cdf of } \xi_T.
\]
Optimal Portfolio via Quantiles

Let $V(\cdot)$ be non decreasing and law invariant, then if there exists a solution to

$$\max_{X_T} \quad V(X_T),$$

then Problem (2) boils down to searching a quantile

$$\sup_{F^{-1} \mid \mathbb{E}[\xi_T F^{-1}(1-F_{\xi_T}(\xi_T))] = \omega_0} V(F^{-1}(1-F_{\xi_T}(\xi_T))).$$

See e.g., He and Zhou: Optimal portfolio via quantiles, Ma.Fi. 2011, among many other authors who used quantiles to solve portfolio selection problems: Dybvig (1988), Föllmer and Schied (2004), Carlier and Dana (2006), Jin and Zhou (2008) and many more after 2011...
Multivariate Risk Sharing (without a market)

Define for each variable $S$,

$$A_d(S) := \left\{ \mathbf{x} : \sum_{i=1}^{d} X_i = S \right\}$$

Assume that we know how to solve

$$\sup_{\mathbf{x} \in A_d(S)} V(X_1, \ldots, X_d). \tag{3}$$

Denote by

$$(Y_1(S), \ldots, Y_d(S))$$

the optimal solution to (3).
Multivariate Risk Sharing (without a market)
Some examples in the literature

- Borch (1962) when \( V(X_1, \ldots, X_d) = \sum_{i=1}^{d} \mathbb{E}[U_i(X_i)] \).
- Inf convolution of convex risk measures Barieu and El Karoui (2005); for law invariant monetary utility by Jouini, Schachermayer and Touzi (2008).
- ...
Towards a Generalization to the Multivariate Case
(Bernard, De Gennaro, Vanduffel EJOR 2023)

Proposition

Consider an investor with law invariant preferences and who is maximizing her objective function \( V(X_1, \ldots, X_d) \) with a given initial budget \( w_0 \), i.e., \( E\left[\xi_T \sum_{i=1}^{d} X_i\right] = w_0 \). Also, assume that \( V(\cdot) \) is strictly increasing in at least one of the \( d \) components. Then the optimal investment for this investor, when it exists, is multivariate cost-efficient, i.e., it solves

\[
\min_{(X_1, \ldots, X_d) \sim G} E\left[\xi_T \sum_{i=1}^{d} X_i\right],
\]

for some joint distribution \( G \).

(all \( X_i, i = 1, \ldots, d \), share same investment horizon \( T \), so we omit it)
Sufficient Condition for Multivariate Cost-efficiency

**Proposition**

A (multidimensional) payoff is multivariate cost-efficient if

\[ \text{cov}(X_1 + X_2 + \ldots + X_d, \xi_T) \]  (4)

is minimum.

This allows us to build a numerical approximation for the optimal solution of a multivariate cost-efficiency problem.
Quantile Formulation of the Multivariate Portfolio Choice

From **multivariate cost-efficiency**, if a portfolio $X_1^*, X_2^*, .., X_d^*$ is a solution to

$$\sup_{E[\xi^T \sum_i X_i] = \omega_0} V(X_1, ..., X_d)$$

then $\sum X_i^* = F_S^{-1}(1 - F_{\xi^T}(\xi_T))$ where $F_S$ is the cdf of $\sum X_i^*$. 
Quantile Formulation of the Multivariate Portfolio Choice

From **multivariate cost-efficiency**, if a portfolio \( X_1^*, X_2^*, \ldots, X_d^* \) is a solution to

\[
\sup_{\mathbb{E}[\xi_T \sum_i X_i] = \omega_0} V(X_1, \ldots, X_d)
\]

then \( \sum X_i^* = F_S^{-1}(1 - F_{\xi_T}(\xi_T)) \) where \( F_S \) is the cdf of \( \sum X_i^* \).

The optimal portfolio then solves

\[
\sup_{F_S^{-1}} \quad \text{s.t.} \quad \mathbb{E}[\xi_T F_S^{-1}(U_T)] = \omega_0
\]

\[
V \left( Y_1(F_S^{-1}(U_T)), \ldots, Y_d(F_S^{-1}(U_T)) \right)
\]

where \( U_T = 1 - F_{\xi_T}(\xi_T) \).
Numerical approach to solve for $F^{-1}_S$

- **Step 1:** Discretize the problem: $\xi_T$ takes $n$ values

  $\xi_1 > \xi_2 > \ldots > \xi_n$

  $\xi_k := F^{-1}_{\xi_T} \left( \frac{n + 1 - k - 0.5}{n} \right)$, for $k = 1, 2, \ldots, n$.

- **Step 2:** Formalize the optimization within a discrete setting. The goal is to solve for $(s_1, s_2, \ldots, s_n)$.

\[
\max_{(s_1, s_2, \ldots, s_n) \in \mathcal{A}} \tilde{V}(s_1, s_2, \ldots, s_n),
\]

(5a)

in which $s_i := F^{-1}_S \left( \frac{i}{n+1} \right)$ and the admissible set $\mathcal{A}$ is

\[
\mathcal{A} = \left\{ (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n \mid \sum_{j=1}^{n} \frac{1}{n} [\xi_j s_j] = \omega_0 \ (\text{budget}) \right\}.
\]
Numerical approach of the portfolio choice via quantiles

- **Step 3:** Translate the fact that for the optimal solution $S$ and $\xi_T$ are anti-monotonic. To do so,

$$\xi_1 > \xi_2 > \ldots > \xi_n \hspace{1cm} \text{and} \hspace{1cm} s_1 \leq s_2 \leq \ldots \leq s_n$$

$s_i$ is an increasing sequence over the states translates in

$$s_1 = z_1 \leq s_2 = z_1 + z_2 \leq \ldots \leq z_1 + z_2 + \ldots + z_n = s_n$$

where the increasing constraint becomes simply $z_i \geq 0$

$$\max_{(z_1, z_2, \ldots, z_n) \in \tilde{A}} \tilde{V} (z_1, z_2, \ldots, z_n), \quad (6a)$$

$$\tilde{A} = \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{R}_+^n \mid \sum_{i=1}^n \xi_i z_i = \omega_0 \hspace{1cm} \text{(budget constraint)} \right\}.$$  

$\Rightarrow$ Solving the above optimization requires using a solver for $n$ dimensions.
Convergence and accuracy of the algorithm

- Need a large $n$... impossible to solve without a good starting guess!
- Trick: Start with very small $n$ and then use this solution as the starting value for the next step with $2n$ discretizations.

**Figure:** Diagram of the algorithm
Two Examples of Explicit Multivariate Portfolios

1. **Expected multivariate utility**: a sum of expected utility
2. **Multivariate Yaari dual theory of choice**: a sum of distorted expectations

Both problems can be solved explicitly and allow us to check that our numerical approach provides accurate solutions.
Example in a Bivariante Expected Utility (theory)

Define $U_{a_1}, U_{a_2}$ are univariate exponential utility functions as

$$U_{a_i}(x) = -e^{-a_i x}, \quad i = 1, 2$$

and $a_1, a_2, \nu_1, \nu_2 > 0$.

**Proposition:** The optimal solutions $X_1^*$ and $X_2^*$ to the problem

$$\max_{(X_1, X_2) \in A} \mathbb{E} \left[ \nu_1 U_{a_1}(X_1) + \nu_2 U_{a_2}(X_2) \right], \quad (7)$$

with $A := \{(X_1, X_2) : \mathbb{E} [\xi_T (X_1 + X_2)] = w_0\}$ are given by

$$\begin{pmatrix} X_1^* \\ X_2^* \end{pmatrix} = \begin{pmatrix} w_0 \lambda^* e^{rT} - \frac{1}{a_1} \left( rT - \frac{\theta^2 T}{2} \right) - \frac{\ln(\xi_T)}{a_1} \\ w_0 (1 - \lambda^*) e^{rT} - \frac{1}{a_2} \left( rT - \frac{\theta^2 T}{2} \right) - \frac{\ln(\xi_T)}{a_2} \end{pmatrix}, \quad (8)$$

with $\lambda^* = \frac{\ln(\frac{\nu_1 a_1}{\nu_2 a_2}) + a_2 w_0 e^{rT}}{(a_1 + a_2) w_0 e^{rT}}$. 
Example in a Bivariate Expected Utility (numerical)

For any variable $S$, define $A_2(S) := \{\mathbf{X} : X_1 + X_2 = S\}$

$$\sup_{\mathbf{X} \in A_2(S)} -v_1 e^{-a_1 X_1} - v_2 e^{-a_2 X_2}$$

(9)

- The optimal bivariate risk sharing rule without a market (solving (9) for any $S$)

$$X_1 = Y_1(S) = a + bS \quad \text{and} \quad X_2 = Y_2(S) = -a + (1 - b)S$$

- Numerical solver to approximate the distribution of $S$. Trick: do a very rough discretization with say $n = 10$, and then solve, and then multiply by 2 the number of discretization points using the previous solution as initial condition... etc
Example in a Bivariate Expected Utility

\[ a_1 = 0.8, \ a_2 = 0.2, \ v_1 = 0.3, \ v_2 = 0.7 \]
Illustration of the convergence
\[ a_1 = 0.8 \ , \ a_2 = 0.2 \ , \ v_1 = 0.3 \ , \ v_2 = 0.7 \]

RAE: relative absolute error (RAE) for the objective function between the solution obtained numerically and the explicit solution.
Another example: Yaari Dual Theory of Choice

An agent with payoff $X_T$ maximizes the distorted expectation (Yaari utility). So the 1-d portfolio choice problem writes

$$\sup_{F_{X_T}^{-1}} \text{s.t. } \mathbb{E}\left[\xi_T F_{X_T}^{-1}(1-F_{\xi_T}(\xi_T))\right] = w_0 \int_0^1 h(u) F_{X_T}^{-1}(u) \, du, \quad (10)$$

in which $h$ is the weighting function; $h(u) := g'(1-u)$ where $g$ is the distortion function.
An example of distorted expectation: RVaR

Let \((\alpha, \beta) \in [0, 1]^2\) be such that \(\alpha \leq \beta\). The **Range Value-at-Risk** (RVaR) is then defined as

\[
\text{RVaR}_{\alpha, \beta}(X) = \begin{cases} 
\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \text{VaR}_u(X) \, du & \text{if } \beta > \alpha \\
\text{VaR}_\alpha(X) & \text{if } \beta = \alpha.
\end{cases}
\]

(Cont, Deguest, Scandolo (2010)).
An example of distorted expectation: RVaR

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\text{VaR}_\alpha(X) & \text{if } \beta = \alpha.
\end{cases}
\]

(Cont, Deguest, Scandolo (2010)).

In the RVaR context,

\[
g(u) = \min \left\{ \max \left\{ \frac{u + \beta - 1}{\beta - \alpha}, 0 \right\}, 1 \right\}
\quad \text{and} \quad
h(u) = \frac{1}{\beta - \alpha} 1_{(\alpha, \beta]}(u)
\]
Multivariate Yaari Dual Theory of Choice

We consider as objective a sum of distorted expectations (Yaari’s expectation).

\[ V(X_1, X_2, \ldots, X_d) = \sum_{i=1}^{d} \rho_{g_i}(X_i) \]

where

\[ \rho_{g_i}(X_i) = \int_{0}^{1} h_i(u) F_{X_i}^{-1}(u) \, du \]

in which \( h_i \) is the weighting function; \( h_i(u) := g_i'(1 - u) \) where \( g_i \) is the distortion function.
Example: Multivariate Portfolio Problem

The multivariate portfolio choice problem under study writes as

$$\sup_{(X_1, X_2, \ldots, X_d) \in A} \sum_{i=1}^{d} \rho g_i(X_i),$$

(11)

where the admissible set $A$ is

$$A = \left\{ (X_1, X_2, \ldots, X_d) \text{ s.t. } X_i \geq 0, \ \mathbb{E} \left[ \xi^T \sum_{i=1}^{d} X_i \right] = w_0 \right\},$$

and $w_0 > 0$ denotes the total budget that must be allocated in $d$ dimensions.
Explicit solution for the Yaari investor \((d = 1)\)

The Yaari Ratio \(YR(c), \forall c > 0\), is defined as

\[ YR(c) = \frac{g(p(c))}{q(c)}, \]

where \(p(c) = \mathbb{P}(\xi_T < c)\) and \(q(c) = \mathbb{E}[\xi_T 1_{\xi_T < c}] e^{rT} \).

**Theorem: Boudt-Dragun-Vanduffel (2022) or He-Jiang (2021):**

The **optimal solution** to the problem (10) is explicit.

1. \(X_T^* = w_0 e^{rT}\) when \(\sup_{c > 0} YR(c) \leq 1\);

2. otherwise, when \(\sup_{c > 0} YR(c) > 1\) and the supremum is attained, it is

\[ X_T^* = \frac{w_0}{q(c^*)} e^{rT} 1_{\xi_T < c^*}, \quad c^* = \arg\max_{c > 0} YR(c). \]
Explicit solution \((d = 2)\)

An example with the following parameters:

- For payoff \(X_1\): \(\alpha_1 = 0.65\), \(\beta_1 = 0.75\), and \(\text{max YR} = 3.58\) with \(c^* = 0.89\);
- For payoff \(X_2\): \(\alpha_2 = 0.6\), \(\beta_2 = 0.9\), and \(\text{max YR} = 3.07\) with \(c^* = 0.92\).

For each unit of budget invested, \(X_1\) is always better than \(X_2\).
Explicit solution (Example with sum of $d = 2$ RVaRs)

$$\sup_{(X_1, X_2) \in A} \text{RVaR}_{\alpha_1, \beta_1}(X_1) + \text{RVaR}_{\alpha_2, \beta_2}(X_2),$$

where $A = \{(X_1, X_2) \in \mathcal{X}_+^d \text{ s.t. } \mathbb{E}[\xi_T(X_1 + X_2)] = w_0\}$. 

- Extreme risk sharing: concentration of payoff in one participant;
- No benefit of investing in payoff $X_2$.
- Digital option for $X_1^*$;
- Nothing for $X_2^*$. 

Carole Bernard
**Proposition: Explicit MV portfolio with sum of Yaari utilities**

Let $Z_i^*$ be the solution to

$$\sup_{Z \in \mathcal{X}_+ / \mathbb{E}[\xi^T Z] = \omega_0} \rho_{g_i}(Z).$$

Assuming that the MV problem has a solution $(X_1^*, ... X_d^*)$ then there are two cases:

- If there exists $i_0$ such that $\rho_{g_{i_0}}(Z_{i_0}^*) > \rho_{g_i}(Z_i^*)$ for all $i \neq i_0$ then the optimal solution is unique and is such that $X_{i_0}^* = Z_{i_0}^*$ and $X_i^* = 0$ for all $i \neq i_0$.

- Otherwise, define $R := \max \rho_{g_i}(Z_i^*)$ let $\mathcal{I} = \{i : \rho_{g_i}(Z_i^*) = R\}$ then there is an infinite number of solutions such that for all $i \not\in \mathcal{I}$, $X_i^* = 0$ and for all $i \in \mathcal{I}$, $X_i^* = k_i Z_i^*$ where $k_i \geq 0$ are such that $\sum_{i \in \mathcal{I}} k_i = 1$ (so that the global budget constraint holds $\sum_{i=1}^d \mathbb{E}[\xi^T X_i^*] = \omega_0$).
GNum approach

General case:

\[
\sup_{(x_{i1}, x_{i2}, \ldots, x_{i(d-1)}, z_i) \in A'} \frac{1}{n} \sum_{i=1}^{n} V \left( x_{i1}, x_{i2}, \ldots, x_{i(d-1)}, \sum_{\ell=1}^{i} z_\ell - \sum_{k=1}^{d-1} x_{ik} \right)
\]

where the admissible set \( A' \) is given by

\[
A' = \left\{ (x_{i1}, x_{i2}, \ldots, x_{i(d-1)}, z_i) \in (\mathbb{R}_+)^d, \mid \sum_{i=1}^{n} \zeta_i z_i = w_0 \right\}
\]

and \( \zeta_i = \frac{1}{n} \sum_{k=i}^{n} \xi_k \), for \( i = 1, \ldots, n \), where \( \xi_1 > \xi_2 > \ldots > \xi_n \).

Start with very small value for \( n \), e.g. \( n = 5 \).
GNum approach

General case:

\[
\sup_{(x_{i1}, x_{i2}, \ldots, x_{i(d-1)}, z_i) \in \mathcal{A}'} \frac{1}{n} \sum_{i=1}^{n} V \left( x_{i1}, x_{i2}, \ldots, x_{i(d-1)}, \sum_{\ell=1}^{i} z_\ell - \sum_{k=1}^{d-1} x_{ik} \right)
\]

where the admissible set \( \mathcal{A}' \) is given by

\[
\mathcal{A}' = \left\{ (x_{i1}, x_{i2}, \ldots, x_{i(d-1)}, z_i) \in (\mathbb{R}_+)^d \mid \sum_{i=1}^{n} \zeta_i z_i = w_0 \right\}
\]

and \( \zeta_i = \frac{1}{n} \sum_{k=i}^{n} \xi_k \), for \( i = 1, \ldots, n \), where \( \xi_1 > \xi_2 > \ldots > \xi_n \).

Start with very small value for \( n \), e.g. \( n = 5 \).

We are looking for \((x_{i1}, x_{i2})\) for \( i = 1, \ldots, n \). Define \( s_i = x_{i1} + x_{i2} \) and \( \tilde{Z} \) such that \( s_i = \sum_{\ell=1}^{i} z_i \) to ensure multivariate cost-efficiency:

\[
\sup_{(x_{i1}, z_i) \in \mathcal{A}'} \text{RVaR}_{\alpha_1, \beta_1}(\tilde{X}_1) + \text{RVaR}_{\alpha_2, \beta_2}(\tilde{S} - \tilde{X}_1).
\]
Figure 3.21: Optimal portfolios in the case of a sum of two RVaR using GNum approach. Input parameters: $\alpha_1 = 0.65$, $\beta_1 = 0.75$, $\alpha_2 = 0.6$, $\beta_2 = 0.9$, $\mu = 0.05$, $r = 0.01$, $\sigma = 0.2$, $T = 1$, and $n_5 = 640$. 
Conclusions, Current & Future Work

► Natural extension of cost-efficiency to a multivariate setting
► Solving a MV portfolio amounts to solve a MV risk sharing problem and search for a one-dimensional quantile.
► Explicit multivariate portfolio for the supconvolution of Distorted expectations, including RVaR as a special case
► An extension to cost-efficiency in incomplete markets. Project with Stephan Sturm.

Do not hesitate to contact me to get updated working papers!
Thank you for listening!