Multivariate Portfolio Choice via Quantiles

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joint work with
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Outline of the Talk

1. Optimal Financial Decision Making
   - Role of cost-efficiency
   - Quantile Approach
   - Towards a generalisation to the multivariate case...
2. Optimal Multivariate Financial Decision Making
   - "Multivariate" cost-efficiency - Characterization of optimum
   - Reduction to a one-dimensional problem
   - Numerical approximation
3. Multivariate Risk Sharing via Quantile Approach
   - Theoretical elements
   - Example with a bivariate expected utility
   - Example with a multivariate Yaari investor

Cost-efficiency

- A portfolio/cash-flow/consumption with final payoff X_T **(consumption only at time T)**.
- A complete market
- Initial cost of X_T is given by $\mathbf{x_0} = \mathbf{c}(\mathbf{X_T}) = \mathbb{E}[\xi_\mathbf{T}\mathbf{X_T}]$.

A strategy $X_T^\star$ (or a payoff) with cdf F is cost-efficient

if any other strategy that generates the same distribution F at the time horizon T costs at least as much, i.e., if it solves

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

Explicit Representation of Cost-efficient Payoffs

Theorem

Consider the cost-efficiency problem:

$$\min_{\{X_T \mid X_T \sim F\}} \mathbb{E}[\xi_T X_T]$$

Assume ξ_T is continuously distributed, *then the optimal strategy is*

$$X_T^\star = F^{-1}\left(1 - F_\xi\left(\xi_T\right)\right).$$

Note that $X_T^\star \sim F$ and $X_T^\star$ is a.s. **unique** *solution.*

Intuition of the proof: $\frac{\mathbb{E}[\xi_T X_T] - \mathbb{E}[\xi_T]\mu_F}{std(\xi_T)\sigma_F} = corr(\xi_T, X_T)$

Cost-efficiency & Portfolio Choice (General preferences)

$V(\cdot)$ **denotes the objective function** of the agent to maximize (Expected utility, Value-at-Risk, Cumulative Prospect Theory...).

$$\max_{X_T \mid \mathbb{E}[\xi_T X_T]=\omega_0} V(X_T). \tag{1}$$

Preferences $V(\cdot)$ **are assumed to be**

- **non-decreasing**: $X_T \geqslant Y_T \ a.s. \Rightarrow V(X_T) \geqslant V(Y_T)$
- **law-invariant**: $X_T =_d Y_T \Rightarrow V(X_T) = V(Y_T)$

Equivalently, $V(\cdot)$ respects **First-order stochastic dominance**

Theorem: Optimal strategies are cost-efficient

If an optimum $X_T^\star$ of (1) exists, let F be its cdf. Then, $X_T^\star$ is the cheapest way (cost-efficient) to achieve F at T, i.e.
$X_T^\star = F^{-1}(1 - F_\xi(\xi_T))$ where F_ξ is the cdf of ξ_T.

Optimal Portfolio via Quantiles

Let $V(\cdot)$ be **non decreasing** and **law invariant**, then if there exists a solution to

$$\max_{X_T \mid \mathbb{E}[\xi_T X_T]=\omega_0} V(X_T), \tag{2}$$

then Problem (2) boils down to searching a **quantile**

$$\sup_{F^{-1} \mid \mathbb{E}\left[\xi_T F^{-1}\left(1-F_{\xi_T}(\xi_T)\right)\right]=\omega_0} V\left(F^{-1}\left(1-F_{\xi_T}(\xi_T)\right)\right).$$

See e.g., He and Zhou: *Optimal portfolio via quantiles*, Ma.Fi. 2011, among many other authors who used quantiles to solve portfolio selection problems: Dybvig (1988), Föllmer and Schied (2004), Carlier and Dana (2006), Jin and Zhou (2008) and many more after 2011...

Multivariate Risk Sharing (without a market)

Define for each variable S,

$$A_d(S) := \left\{ \mathbf{X} \ : \ \sum_{i=1}^{d} X_i = S \right\}$$

Assume that we **know** how to solve

$$\sup_{\mathbf{X} \in A_d(S)} V(X_1, ..., X_d). \tag{3}$$

Denote by

$$(Y_1(S), ..., Y_d(S))$$

the optimal solution to (3).

Multivariate Risk Sharing (without a market)
Some examples in the literature

- Borch (1962) when $V(X_1, ..., X_d) = \sum_{i=1}^{d} \mathbb{E}[U_i(X_i)]$.
- Inf convolution of convex risk measures Barieu and El Karoui (2005); for law invariant monetary utility by Jouini, Schachermayer and Touzi (2008)
- Some further generalizations by Acciaio (2007), Filipovic and Svindland (2008) and Carlier, Dana, and Galichon (2012).
- Inf convolution of quantile risk measures: Embrechts, Liu and Wang (2018).
- ...

Towards a Generalization to the Multivariate Case (Bernard, De Gennaro, Vanduffel EJOR 2023)

Proposition

Consider an investor with **law invariant** preferences and who is maximizing her objective function $V(X_1, \ldots, X_d)$ with a given initial budget w_0, i.e., $\mathbb{E}\left[\xi_T \sum_{i=1}^d X_i\right] = w_0$. Also, assume that $V(\cdot)$ is **strictly increasing in at least one of the** d **components**. Then the optimal investment for this investor, when it exists, is **multivariate cost-efficient**, i.e., it solves

$$\min_{(X_1,\ldots,X_d)\sim G} \mathbb{E}\left[\xi_T \sum_{i=1}^d X_i\right],$$

for some joint distribution G.

(all $X_i, i = 1, \ldots, d$, share same investment horizon T, so we omit it)

Sufficient Condition for Multivariate Cost-efficiency

Proposition

A (multidimensional) payoff is multivariate cost-efficient if

$$\text{cov}(X_1 + X_2 + ... + X_d, \xi_{\mathbf{T}}) \quad (4)$$

is minimum.

This allows us to build a numerical approximation for the optimal solution of a multivariate cost-efficiency problem.

Quantile Formulation of the Multivariate Portfolio Choice

From **multivariate cost-efficiency**, if a portfolio $X_1^*, X_2^*, .., X_d^*$ is a solution to

$$\sup_{\mathbb{E}[\xi_T \sum_i X_i]=\omega_0} V(X_1, ..., X_d)$$

then $\sum X_i^* = F_S^{-1}(1 - F_{\xi_T}(\xi_T))$ where F_S is the cdf of $\sum X_i^*$.

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The optimal portfolio then solves

$$\sup_{F_S^{-1} \ s.t. \ \mathbb{E}[\xi_T F_S^{-1}(U_T)]=\omega_0} V\left(Y_1(F_S^{-1}(U_T)), ..., Y_d(F_S^{-1}(U_T))\right)$$

where $U_T = 1 - F_{\xi_T}(\xi_T)$.

Numerical approach to solve for F_S^{-1}

- Step 1: Discretize the problem: ξ_T takes n values

$$\xi_1 > \xi_2 > ... > \xi_n$$

$$\xi_k := F_{\xi_T}^{-1}\left(\frac{n+1-k-0.5}{n}\right), \quad \text{for } k = 1, 2, \dots, n.$$

- Step 2: Formalize the optimization within a discrete setting. The goal is to solve for $(s_1, s_2, ..., s_n)$.

$$\max_{(s_1,s_2,...,s_n)\in\mathcal{A}} \widetilde{V}(s_1, s_2, ..., s_n), \tag{5a}$$

in which $s_i := F_S^{-1}\left(\frac{i}{n+1}\right)$ and the admissible set $\mathcal{A}$ is

$$\mathcal{A} = \left\{(s_1, s_2, ..., s_n) \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^n \frac{1}{n}[\xi_j s_j] = \omega_0 \;\; \text{(budget)} \right\}.$$

Numerical approach of the portfolio choice via quantiles

- Step 3: Translate the fact that for the optimal solution S and ξ_T are anti-monotonic. To do so,

$$\xi_1 > \xi_2 > ... > \xi_n \quad and \quad s_1 \leqslant s_2 \leqslant ... \leqslant s_n$$

s_i is an increasing sequence over the states translates in

$$s_1 = z_1 \leqslant s_2 = z_1 + z_2 \leqslant ... \leqslant z_1 + z_2 + ... + z_n = s_n$$

where the increasing constraint becomes simply $z_i \geqslant 0$

$$\max_{(z_1, z_2, ..., z_n) \in \widetilde{\mathcal{A}}} \widetilde{V}\left(z_1, z_2, ..., z_n\right), \tag{6a}$$

$\widetilde{\mathcal{A}} = \left\{ (z_1, z_2, ..., z_n) \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^n \zeta_i z_i = \omega_0 \;\; \text{(budget constraint)} \;\right\}.$

⇒ Solving the above optimization requires using a solver for n dimensions.

Convergence and accuracy of the algorithm

- Need a large n... impossible to solve without a good starting guess!
- Trick: Start with very small n and then use this solution as the starting value for the next step with $2n$ discretizations.

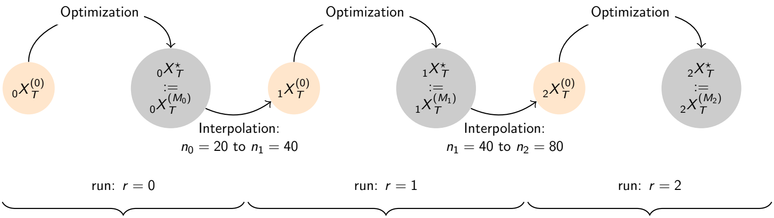


Figure: Diagram of the algorithm

Two Examples of Explicit Multivariate Portfolios

1. **Expected multivariate utility** : a sum of expected utility
2. **Multivariate Yaari dual theory of choice** : a sum of distorted expectations

Both problems can be solved explicitly and allow us to check that our numerical approach provides accurate solutions.

Example in a Bivariate Expected Utility (theory)

Define U_{a_1}, U_{a_2} are univariate exponential utility functions as

$$U_{a_i}(x) = -e^{-a_i x}, \quad i = 1, 2$$

and $a_1, a_2, v_1, v_2 > 0$.

Proposition: The optimal solutions $X_1^\star$ and $X_2^\star$ to the problem

$$\max_{(X_1, X_2) \in \mathcal{A}} \mathbb{E}\left[v_1 U_{a_1}(X_1) + v_2 U_{a_2}(X_2)\right], \tag{7}$$

with $\mathcal{A} := \{(X_1, X_2) : \mathbb{E}\left[\xi_T (X_1 + X_2)\right] = w_0\}$ are given by

$$\begin{pmatrix} X_1^\star \\ X_2^\star \end{pmatrix} = \begin{pmatrix} w_0 \lambda^\star e^{rT} - \frac{1}{a_1}\left(rT - \frac{\theta^2 T}{2}\right) - \frac{\ln(\xi_T)}{a_1} \\ w_0 (1 - \lambda^\star) e^{rT} - \frac{1}{a_2}\left(rT - \frac{\theta^2 T}{2}\right) - \frac{\ln(\xi_T)}{a_2} \end{pmatrix} \tag{8}$$

with $\lambda^\star = \frac{\ln\left(\frac{v_1 a_1}{v_2 a_2}\right) + a_2 w_0 e^{rT}}{(a_1 + a_2) w_0 e^{rT}}$.

Example in a Bivariate Expected Utility (numerical)

For any variable S, define $A_2(S) := \{\mathbf{X} \;:\; X_1 + X_2 = S\}$

$$\sup_{\mathbf{X}\in A_2(S)} -v_1 e^{-a_1 X_1} - v_2 e^{-a_2 X_2} \tag{9}$$

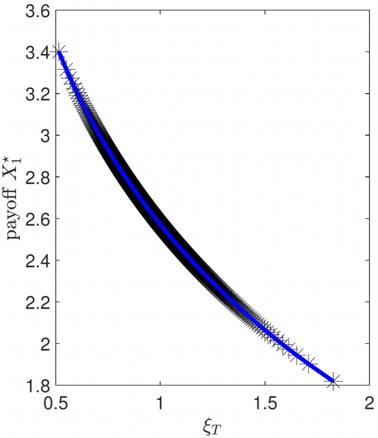
- The optimal bivariate risk sharing rule without a market (solving (9) for any S)

$$X_1 = Y_1(S) = a + bS \quad \text{and} \quad X_2 = Y_2(S) = -a + (1-b)S$$

- **Numerical solver to approximate the distribution of S.** Trick: do a very rough discretization with say $n = 10$, and then solve, and then multiply by 2 the number of discretization points using the previous solution as initial condition... etc

Example in a Bivariate Expected Utility

$a_1 = 0.8$, $a_2 = 0.2$, $v_1 = 0.3$, $v_2 = 0.7$



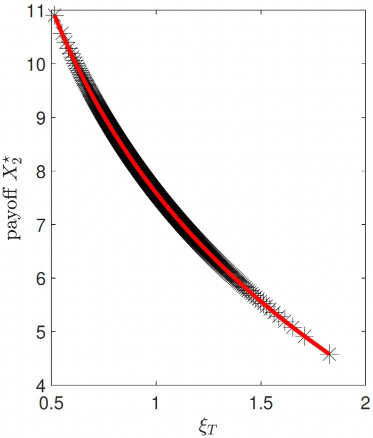
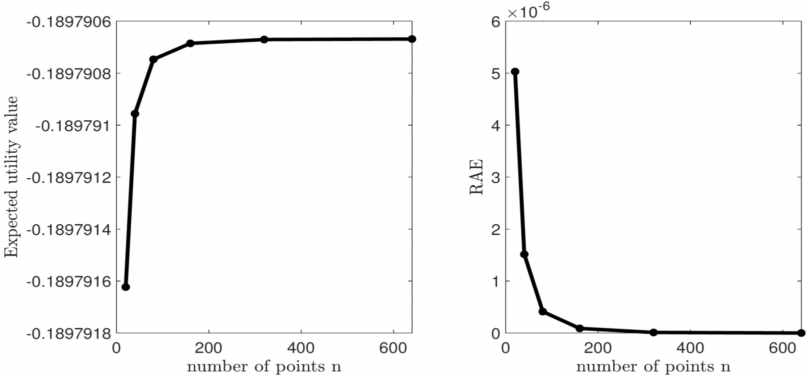


Illustration of the convergence

$a_1 = 0.8$, $a_2 = 0.2$, $v_1 = 0.3$, $v_2 = 0.7$



RAE: relative absolute error (RAE) for the objective function between the solution obtained numerically and the explicit solution.

Another example: Yaari Dual Theory of Choice

An agent with payoff X_T maximizes the distorted expectation (Yaari utility). So the 1-d portfolio choice problem writes

$$\sup_{F_{X_T}^{-1} \text{ s.t. } \mathbb{E}\left[\xi_T F_{X_T}^{-1}(1-F_{\xi_T}(\xi_T))\right]=w_0} \int_0^1 h(u)\, F_{X_T}^{-1}(u)\, \mathrm{d}u, \tag{10}$$

in which h is the weighting function; $h(u) := g'(1-u)$ where g is the distortion function.

An example of distorted expectation: RVaR

Let $(\alpha, \beta) \in [0,1]^2$ be such that $\alpha \leqslant \beta$. The **Range Value-at-Risk** (RVaR) is then defined as

$$\text{RVaR}_{\alpha,\beta}(X) = \begin{cases} \dfrac{1}{\beta-\alpha}\displaystyle\int_\alpha^\beta \text{VaR}_u(X)\,\mathrm{d}u & \text{if } \beta > \alpha \\ \text{VaR}_\alpha(X) & \text{if } \beta = \alpha. \end{cases}$$

(Cont, Deguest, Scandolo (2010)).

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(Cont, Deguest, Scandolo (2010)).

In the **RVaR** context,

$$g(u) = \min\left\{\max\left\{\frac{u+\beta-1}{\beta-\alpha}, 0\right\}, 1\right\} \text{ and } h(u) = \frac{1}{\beta-\alpha}\mathbb{1}_{(\alpha,\beta]}(u)$$

Multivariate Yaari Dual Theory of Choice

We consider as objective a **sum of distorted expectations** (Yaari's expectation).

$$V(X_1, X_2, ..., X_d) = \sum_{i=1}^{d} \rho_{g_i}(X_i)$$

where

$$\rho_{g_i}(X_i) = \int_0^1 h_i(u)\, F_{X_i}^{-1}(u)\,\mathrm{d}u$$

in which h_i is the weighting function; $h_i(u) := g_i'(1-u)$ where g_i is the distortion function.

Sum of Distorted Expectations with a Financial Market

Example: Multivariate Portfolio Problem

The multivariate portfolio choice problem under study writes as

$$\sup_{(X_1,X_2,\ldots,X_d)\in\mathcal{A}} \sum_{i=1}^{d} \rho_{g_i}(X_i), \tag{11}$$

where the admissible set $\mathcal{A}$ is

$$\mathcal{A} = \left\{ (X_1, X_2, \ldots, X_d) \text{ s.t. } X_i \geqslant 0,\ \mathbb{E}\left[\xi_T \sum_{i=1}^{d} X_i\right] = w_0 \right\},$$

and $w_0 > 0$ denotes the total budget that must be allocated in d dimensions.

Explicit solution for the Yaari investor ($d = 1$)

The Yaari Ratio $\text{YR}(c)$, $\forall c > 0$, is defined as

$$\text{YR}(c) = \frac{g(p(c))}{q(c)},$$

where $p(c) = \mathbb{P}(\xi_T < c)$ and $q(c) = \mathbb{E}[\xi_T \mathbb{1}_{\xi_T < c}]e^{rT}$.

Theorem: Boudt-Dragun-Vanduffel (2022) or He-Jiang (2021):

The optimal solution to the problem (10) is explicit.

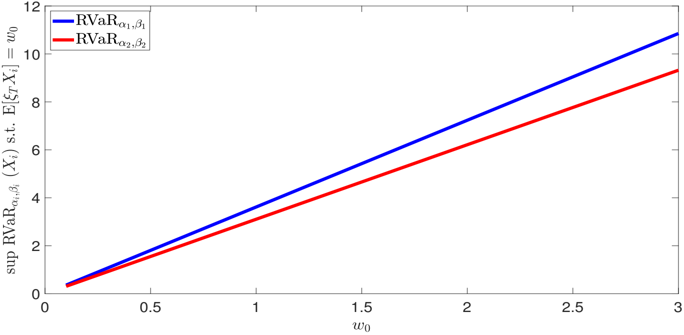
1. $X_T^\star = w_0 e^{rT}$ when $\sup_{c>0} \text{YR}(c) \leqslant 1$;
2. otherwise, when $\sup_{c>0} \text{YR}(c) > 1$ and the supremum is attained, it is

$$X_T^\star = \frac{w_0}{q(c^\star)} e^{rT} \mathbb{1}_{\xi_T < c^\star}, \quad c^* = \arg\max_{c>0} YR(c).$$

Explicit solution ($d = 2$)

An example with the following parameters:

- For payoff X_1: $\alpha_1 = 0.65$, $\beta_1 = 0.75$, and $\max \mathrm{YR} = 3.58$ with $c^\star = 0.89$;
- For payoff X_2: $\alpha_2 = 0.6$, $\beta_2 = 0.9$, and $\max \mathrm{YR} = 3.07$ with $c^\star = 0.92$.

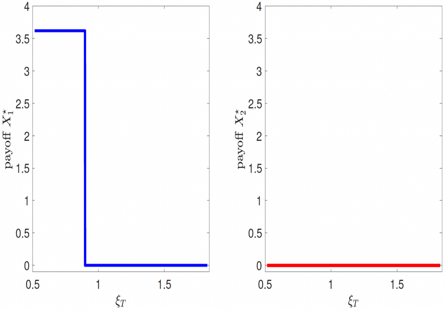


For each unit of budget invested, X_1 is always better than X_2.

Explicit solution (Example with sum of $d = 2$ RVaRs)

$$\sup_{(X_1,X_2)\in\mathcal{A}} \mathrm{RVaR}_{\alpha_1,\beta_1}(X_1) + \mathrm{RVaR}_{\alpha_2,\beta_2}(X_2),$$

where $\mathcal{A} = \left\{(X_1, X_2) \in \mathcal{X}_+^d \text{ s.t. } \mathbb{E}\left[\xi_T(X_1 + X_2)\right] = w_0\right\}.$



- Extreme risk sharing: concentration of payoff in one participant;
- No benefit of investing in payoff X_2.
- Digital option for $X_1^\star$;
- Nothing for $X_2^\star$.

Proposition: Explicit MV portfolio with sum of Yaari utilities

Let Z_i^* be the solution to

$$\sup_{Z\in\mathscr{X}_+/\mathbb{E}[\xi_T Z]=\omega_0} \rho_{g_i}(Z).$$

Assuming that the MV problem has a solution $(X_1^*, ...X_d^*)$ then there are two cases:

- If there exists i_0 such that $\rho_{g_{i_0}}(Z_{i_0}^*) > \rho_{g_i}(Z_i^*)$ for all $i \neq i_0$ then the optimal solution is unique and is such that $X_{i_0}^* = Z_{i_0}^*$ and $X_i^* = 0$ for all $i \neq i_0$.
- Otherwise, define $R := \max \rho_{g_i}(Z_i^*)$ let $\mathcal{I} = \{i : \rho_{g_i}(Z_i^*) = R\}$ then there is an infinite number of solutions such that for all $i \notin \mathcal{I}$, $X_i^* = 0$ and for all $i \in \mathcal{I}$, $X_i^* = k_i Z_i^*$ where $k_i \geqslant 0$ are such that $\sum_{i\in\mathcal{I}} k_i = 1$ (so that the global budget constraint holds $\sum_{i=1}^d \mathbb{E}[\xi_T X_i^*] = \omega_0$).

GNum approach

General case:

$$\sup_{\left(x_{i1},x_{i2},\ldots,x_{i(d-1)},z_i\right)_{i=1,\ldots,n}\in\mathcal{A}'} \frac{1}{n}\sum_{i=1}^{n} V\left(x_{i1},x_{i2},\ldots,x_{i(d-1)},\sum_{\ell=1}^{i} z_\ell - \sum_{k=1}^{d-1} x_{ik}\right)$$

where the admissible set $\mathcal{A}'$ is given by

$$\mathcal{A}' = \left\{ \left(x_{i1},x_{i2},\ldots,x_{i(d-1)},z_i\right) \in \left(\mathbb{R}_+\right)^d, \mid \ \sum_{i=1}^n \zeta_i z_i = w_0 \quad \right\}$$

and $\zeta_i = \frac{1}{n}\sum_{k=i}^{n}\xi_k$, for $i = 1,\ldots,n$, where $\xi_1 > \xi_2 > ... > \xi_n$.

Start with very small value for n, e.g. $n = 5$.

GNum approach

General case:

$$\sup_{\left(x_{i1},x_{i2},\ldots,x_{i(d-1)},z_i\right)_{i=1,\ldots,n}\in\mathcal{A}'} \frac{1}{n}\sum_{i=1}^{n} V\left(x_{i1},x_{i2},\ldots,x_{i(d-1)},\sum_{\ell=1}^{i} z_\ell-\sum_{k=1}^{d-1}x_{ik}\right)$$

where the admissible set $\mathcal{A}'$ is given by

$$\mathcal{A}'=\left\{\left(x_{i1},x_{i2},\ldots,x_{i(d-1)},z_i\right)\in(\mathbb{R}_+)^d,\mid\ \sum_{i=1}^n\zeta_i z_i=w_0\quad\right\}$$

and $\zeta_i=\frac{1}{n}\sum_{k=i}^n\xi_k$, for $i=1,\ldots,n$, where $\xi_1>\xi_2>...>\xi_n$.

Start with very small value for n, e.g. $n=5$.

We are looking for (x_{i1},x_{i2}) for $i=1,...,n$. Define $s_i=x_{i1}+x_{i2}$ and $\vec{Z}$ such that $s_i=\sum_{\ell=1}^i z_i$ to ensure multivariate cost-efficiency:

$$\sup_{(x_{i1},z_i)_{i=1,\ldots,n}\in\mathcal{A}'} RVaR_{\alpha_1,\beta_1}(\vec{X}_1)+RVaR_{\alpha_2,\beta_2}(\vec{S}-\vec{X}_1).$$

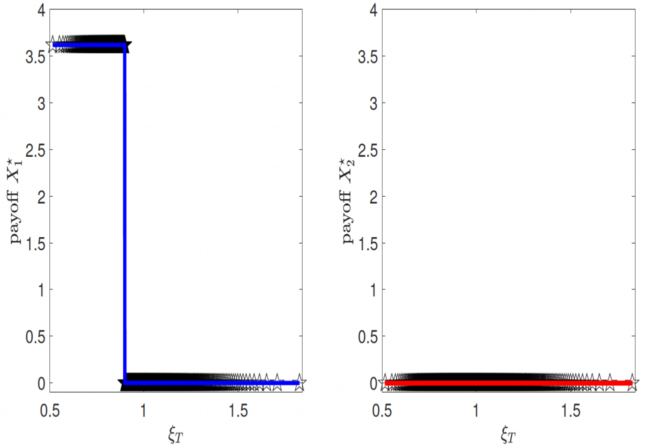


Figure 3.21: Optimal portfolios in the case of a sum of two RVaR using GNum approach. Input parameters: $\alpha_1 = 0.65$, $\beta_1 = 0.75$, $\alpha_2 = 0.6$, $\beta_2 = 0.9$, $\mu = 0.05$, $r = 0.01$, $\sigma = 0.2$, $T = 1$, and $n_5 = 640$.

Conclusions, Current & Future Work

- Natural extension of cost-efficiency to a multivariate setting
- Solving a **MV portfolio** amounts to solve a **MV risk sharing** problem and search for a **one-dimensional quantile**.
- **Explicit multivariate portfolio for the supconvolution of Distorted expectations**, including RVaR as a special case
- An extension to cost-efficiency under **ambiguity**. Project with Gero Junike, Thibaut Lux and Steven Vanduffel forthcoming in *Finance and Stochastics*.
- An extension to cost-efficiency in **incomplete markets**. Project with Stephan Sturm.

Do not hesitate to contact me to get updated working papers!

Thank you for listening !