Signature-based models: theory, calibration, and expansions

Sara Svaluto-Ferro

UNIVERSITÀ di VERONA

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Signatures...why?

Because the (time extended) signature of a continuous semimartingale uniquely determines its path...

...and because every polynomial on the signature has a linear representative.

$\longrightarrow$ If $S_T = F((X_t)_{t\in[0,T]})$ for some continuous map F, then

$$S_T \approx L(\widehat{\mathbb{X}}_T)$$

for some linear map L, where $\widehat{\mathbb{X}}$ denotes the signature of $t \mapsto (t, X_t)$.

$\longrightarrow$ Linear regressions, affine and polynomial technology, and other useful machinery can be applied!

Signature: definition and properties

Signature of a 1 dimensional path of finite variation

The signature $(\mathbb{X}_t)_{t\in[0,T]}$ of a 1-dimensional path $(X_t)_{t\in[0,T]}$ of finite variation is defined as

$$\mathbb{X}_t = (1, \int_0^t 1 dX_{t_1}, \int_0^t \int_0^{t_1} 1 dX_{t_2} dX_{t_1}, \int_0^t \int_0^{t_1} \int_0^{t_2} 1 dX_{t_3} dX_{t_2} dX_{t_1}, \ldots),$$

where the integrals are all Rieman-Stiltjes integrals.

- State space: extended tensor algebra $T((\mathbb{R})) = \{(a_0, a_1, a_2, \ldots) : a_i \in \mathbb{R}\}$.
- Notation: we use $\langle e_\emptyset, \mathbb{X}_t\rangle := 1$ and denote the element of $\mathbb{X}_t$ corresponding to k iterated integrals with respect to X as

$$\langle \underbrace{e_1 \otimes \cdots \otimes e_1}_{k \text{ times}}, \mathbb{X}_t\rangle \quad \text{or} \quad \langle e_1^{\otimes k}, \mathbb{X}_t\rangle \quad \text{or} \quad \langle e_I, \mathbb{X}_t\rangle \text{ for } I := (\underbrace{1, \ldots, 1}_{k \text{ times}})$$

- Observation: the k-th term of the signature, is given by the $(k-1)$-th term of the signature integrated from 0 to t:

$$\int_0^t \langle e_1^{\otimes(k-1)}, \mathbb{X}_s\rangle dX_s = \langle e_1^{\otimes k}, \mathbb{X}_t\rangle$$

- Attention: the signature of $(X_t)_{t\in[0,T]}$ and $(X_t + c)_{t\in[0,T]}$ for $c \in \mathbb{R}$ coincide!

Signature of a 1 dimensional path of finite variation

The signature $(\mathbb{X}_t)_{t\in[0,T]}$ of a 1-dimensional path $(X_t)_{t\in[0,T]}$ of finite variation is the path taking values in $T((\mathbb{R}))$ given by

$$\mathbb{X}_t = (\langle e_\emptyset, \mathbb{X}_t\rangle, \langle e_1, \mathbb{X}_t\rangle, \langle e_1^{\otimes 2}, \mathbb{X}_t\rangle, \langle e_1^{\otimes 3}, \mathbb{X}_t\rangle, \dots),$$

for $\langle e_\emptyset, \mathbb{X}_t\rangle = 1$ and $\langle e_1^{\otimes k}, \mathbb{X}_t\rangle = \int_0^t \langle e_1^{\otimes (k-1)}, \mathbb{X}_s\rangle dX_s$.

Signature of a d dimensional path of finite variation

The signature $(\mathbb{X}_t)_{t\in[0,T]}$ of an $\mathbb{R}^d$-valued path $(X_t^1,\ldots,X_t^d)_{t\in[0,T]}$ of finite variation is defined as

$$\mathbb{X}_t = \left(1, \int_0^t 1dX_{t_1}^1, \ldots, \int_0^t 1dX_{t_1}^d, \int_0^t \int_0^{t_1} 1dX_{t_2}^1 dX_{t_1}^1, \int_0^t \int_0^{t_1} 1dX_{t_2}^1 dX_{t_1}^2 \ldots\right),$$

where the integrals are all Rieman-Stiltjes integrals.

- State space: extended tensor algebra

$$T((\mathbb{R}^d)) = \{(a_0, a_1, \ldots) : a_i \in \underbrace{(\mathbb{R}^d)^{\otimes i}}_{\cong \mathbb{R}^{d^i}\text{,i.e. 1 dim } \forall \text{ iterated integral of deep } i}\}.$$

- Notation: we denote the element of $\mathbb{X}_t$ corresponding to the i_1-th element of X, integrated wrt the i_2-th component of X,..., integrated wrt the i_n-component of X as

$$\langle e_{i_1} \otimes \cdots \otimes e_{i_n}, \mathbb{X}_t\rangle \qquad \text{or} \qquad \langle e_I, \mathbb{X}_t\rangle \text{ for } I = (i_1, \ldots, i_n).$$

⇒ Example: $X_t = (Y_t, Z_t)$, $\langle e_1 \otimes e_2, \mathbb{X}_t\rangle = \int_0^t \int_0^s 1dY_r dZ_s = \int_0^t (Y_s - Y_0)dZ_s$.

- Observation: signature terms can be defined recursively: for $I = (i_1, \ldots, i_n)$ we have

$$\langle e_I, \mathbb{X}_t\rangle = \int_0^t \langle e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}, \mathbb{X}_s\rangle dX_s^{i_n} = \langle e_{i_1} \otimes \cdots \otimes e_{i_n}, \mathbb{X}_s\rangle.$$

Signature of a d dimensional path of finite variation

The signature $(\mathbb{X}_t)_{t\in[0,T]}$ of a d-dimensional path $(X_t)_{t\in[0,T]}$ of finite variation is the path taking values in $T((\mathbb{R}^d))$ given by

$$\mathbb{X}_t = (\langle e_\emptyset, \mathbb{X}_t\rangle, \langle e_1, \mathbb{X}_t\rangle, \ldots, \langle e_d, \mathbb{X}_t\rangle, \langle e_1^{\otimes 2}, \mathbb{X}_t\rangle, \langle e_1 \otimes e_2, \mathbb{X}_t\rangle, \ldots),$$

for $\langle e_\emptyset, \mathbb{X}_t\rangle = 1$ and

$$\langle e_I, \mathbb{X}_t\rangle = \int_0^t \langle e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}, \mathbb{X}_s\rangle dX_s^{i_n},$$

for $I = (i_1, \ldots i_n)$.

Signature of a d dimensional continuous semimartingale

The signature $(\mathbb{X}_t)_{t\in[0,T]}$ of a d-dimensional continuous semimartingale $(X_t)_{t\in[0,T]}$ is the process taking values in $T((\mathbb{R}^d))$ given by

$$\mathbb{X}_t = (\langle e_\emptyset, \mathbb{X}_t\rangle, \langle e_1, \mathbb{X}_t\rangle, \ldots, \langle e_d, \mathbb{X}_t\rangle, \langle e_1^{\otimes 2}, \mathbb{X}_t\rangle, \langle e_1 \otimes e_2, \mathbb{X}_t\rangle, \ldots),$$

for $\langle e_\emptyset, \mathbb{X}_t\rangle = 1$ and

$$\langle e_I, \mathbb{X}_t\rangle = \int_0^t \langle e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}, \mathbb{X}_s\rangle \circ dX_s^{i_n},$$

where $I = (i_1, \ldots i_n)$ and $\circ$ denotes the Stratonivoch integral:

$$\int_0^t Y_t \circ dZ_t = \int_0^t Y_t dZ_t + \frac{1}{2}[Y, Z]_t.$$

The shuffle property or the integration by parts formula

Stratonovich (and Riemann-Stiltjes) integrals satisfy the integration by parts formula:

$$\int_0^t Y_s \circ dZ_s = Y_t Z_t - Z_0 Y_0 - \int_0^t Z_s \circ dY_s.$$

Setting $Y_t = \langle e_I, \mathbb{X}_t \rangle$ and $Z_t = \langle e_J, \mathbb{X}_t \rangle$ this yields

$$\begin{aligned} \langle e_I, \mathbb{X}_t \rangle \langle e_J, \mathbb{X}_t \rangle &= \int_0^t \langle e_I, \mathbb{X}_t \rangle \circ d\langle e_J, \mathbb{X}_t \rangle + \int_0^t \langle e_J, \mathbb{X}_t \rangle \circ d\langle e_I, \mathbb{X}_t \rangle \\ &= \int_0^t \langle e_I, \mathbb{X}_t \rangle \langle e_{J'}, \mathbb{X}_t \rangle \circ dX_t^{j_m} + \int_0^t \langle e_J, \mathbb{X}_t \rangle \langle e_{I'}, \mathbb{X}_t \rangle \circ dX_t^{i_n}, \end{aligned}$$

for $e_I = e_{I'} \otimes e_{i_n}$ and $e_J = e_{J'} \otimes e_{j_m}$.

Defining $e_I \sqcup\!\sqcup\, e_\emptyset = e_\emptyset \sqcup\!\sqcup\, e_I = e_I$ and then recursively

$$e_I \sqcup\!\sqcup\, e_J := (e_I \sqcup\!\sqcup\, e_{J'}) \otimes e_{j_m} + (e_J \sqcup\!\sqcup\, e_{I'}) \otimes e_{i_n}$$

we get

$$\langle e_I, \mathbb{X}_t \rangle \langle e_J, \mathbb{X}_t \rangle = \underbrace{\langle e_I \sqcup\!\sqcup\, e_J, \mathbb{X}_t \rangle}_{\text{linear combination of } \mathbb{X}_t\text{'s elements!}}.$$

Every polynomial in the signature has a linear representation!

Examples examples...

Set for simplicity that $X_0 = 0$.

$$
\begin{aligned}
\langle e_1, \mathbb{X}_t \rangle^2 &= (X_t)^2 \overset{\text{Itô}}{=} 2\int_0^t X_s dX_s + [X]_t = 2\int_0^t X_s \circ dX_s = 2\langle e_1 \otimes e_1, \mathbb{X}_t \rangle \\
&\Rightarrow \quad e_1 \sqcup\!\sqcup\, e_1 = 2 e_1 \otimes e_1 \\
\langle e_1, \mathbb{X}_t \rangle \langle e_2, \mathbb{X}_t \rangle &= X_t^1 X_t^2 \overset{\text{Itô}}{=} \int_0^t X_s^1 dX_s^2 + \int_0^t X_s^2 dX_s^1 + [X^1, X^2]_t \\
&= \int_0^t X_s^1 \circ dX_s^2 + \int_0^t X_s^2 \circ dX_s^1 = \langle e_1 \otimes e_2, \mathbb{X}_t \rangle + \langle e_2 \otimes e_1, \mathbb{X}_t \rangle \\
&\Rightarrow \quad e_1 \sqcup\!\sqcup\, e_2 = e_1 \otimes e_2 + e_2 \otimes e_1 \\
\langle e_1, \mathbb{X}_t \rangle^k &= k! \langle e_1^{\otimes k}, \mathbb{X}_t \rangle \\
&\Rightarrow \quad e_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, e_1 = k! e_1^{\otimes k} \\
&\Rightarrow \quad \langle e_1^{\otimes k}, \mathbb{X}_t \rangle = \frac{(X_t)^k}{k!} \\
&\Rightarrow \quad \text{If } X \text{ is 1-dimensional: } \mathbb{X}_t = \left(1, X_t, \frac{(X_t)^2}{2!}, \frac{(X_t)^3}{3!}, \ldots\right)
\end{aligned}
$$

Examples, examples,...

Example

Set $X_t = t$. Then

$$\mathbb{X}_t = (1, t, \tfrac{t^2}{2}, \tfrac{t^3}{6}, \dots, \tfrac{t^k}{k!}, \dots).$$

Example

Let X be a one dimensional continuous semimartingale with $X_0 = 0$. Then

$$\mathbb{X}_t = (1, X_t, \tfrac{X_t^2}{2}, \tfrac{X_t^3}{6}, \dots, \tfrac{X_t^k}{k!}, \dots).$$

Example

Consider $\widehat{X}_t = (t, X_t)$, where X is a one dimensional continuous semimartingale with $X_0 = 0$. Then

$$\widehat{\mathbb{X}}_t = (1, t, X_t, \tfrac{t^2}{2}, \textstyle\int_0^t s dX_s, \int_0^t X_s ds, \tfrac{X_t^2}{2}, \tfrac{t^3}{6}, \dots).$$

Uniqueness of the time extended signature

...namely: the value $\widehat{\mathbb{X}}_T$ of the signature $\widehat{\mathbb{X}}$ of $\widehat{X}_t := (t, X_t)$ at time T uniquely determines the trajectories of $(X_t - X_0)_{t \in [0,T]}$.

Why? For each k and i

$$\int_0^T (X_s^i - X_0^i) \frac{s^k}{k!} ds$$

can be written as (finite) linear combination of $\widehat{\mathbb{X}}_T$'s components!

Welcome back Markovianity :).

The Chen relation

Set

$$\mathbb{X}_{s,t} = (\langle e_\emptyset, \mathbb{X}_{s,t}\rangle, \langle e_1, \mathbb{X}_{s,t}\rangle, \dots, \langle e_d, \mathbb{X}_{s,t}\rangle, \langle e_1^{\otimes 2}, \mathbb{X}_{s,t}\rangle, \langle e_1 \otimes e_2, \mathbb{X}_{s,t}\rangle, \dots),$$

for $\langle e_\emptyset, \mathbb{X}_{s,t}\rangle = 1$ and

$$\langle e_I, \mathbb{X}_{s,t}\rangle = \int_s^t \langle e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}, \mathbb{X}_{s,r}\rangle \circ dX_r^{i_n},$$

where $I = (i_1, \dots i_n)$ and $\circ$ denotes the Stratonivoch integral.

Lemma (Chen relation)

$$\langle e_I, \mathbb{X}_{0,t}\rangle = \sum_{e_{I_1} \otimes e_{I_2} = e_I} \langle e_{I_1}, \mathbb{X}_{0,s}\rangle \langle e_{I_2}, \mathbb{X}_{s,t}\rangle.$$

Examples examples

Chen relation: $\langle e_I, \mathbb{X}_{0,t}\rangle = \sum_{e_{I_1}\otimes e_{I_2}=e_I}\langle e_{I_1}, \mathbb{X}_{0,s}\rangle\langle e_{I_2}, \mathbb{X}_{s,t}\rangle$.

For $I=(1)$ it reads

$$\underbrace{\langle e_{(1)}, \mathbb{X}_{0,t}\rangle}_{=X_t-X_0} = \underbrace{\langle e_{(1)}, \mathbb{X}_{0,s}\rangle\langle e_{\emptyset}, \mathbb{X}_{s,t}\rangle}_{=(X_s-X_0)} + \underbrace{\langle e_{\emptyset}, \mathbb{X}_{0,s}\rangle\langle e_{(1)}, \mathbb{X}_{s,t}\rangle}_{=(X_t-X_s)}.$$

For $I=(1,1)$ it reads

$$\underbrace{\langle e_{(1,1)}, \mathbb{X}_{0,t}\rangle}_{=\frac{(X_t-X_0)^2}{2}} = \underbrace{\langle e_{(1,1)}, \mathbb{X}_{0,s}\rangle\langle e_{\emptyset}, \mathbb{X}_{s,t}\rangle}_{=\frac{(X_s-X_0)^2}{2}} + \underbrace{\langle e_{(1)}, \mathbb{X}_{0,s}\rangle\langle e_{(1)}, \mathbb{X}_{s,t}\rangle}_{=(X_s-X_0)(X_t-X_s)} + \underbrace{\langle e_{\emptyset}, \mathbb{X}_{0,s}\rangle\langle e_{(1,1)}, \mathbb{X}_{s,t}\rangle}_{=\frac{(X_t-X_s)^2}{2}}.$$

It can also be used for:

$$\mathbb{E}[\langle e_I, \mathbb{X}_{0,t}\rangle|\mathcal{F}_s] = \sum_{e_{I_1}\otimes e_{I_2}=e_I}\langle e_{I_1}, \mathbb{X}_{0,s}\rangle\mathbb{E}[\langle e_{I_2}, \mathbb{X}_{s,t}\rangle|\mathcal{F}_s].$$

If X has independent increments this reduces to

$$\mathbb{E}[\langle e_I, \mathbb{X}_{0,t}\rangle|\mathcal{F}_s] = \sum_{e_{I_1}\otimes e_{I_2}=e_I}\langle e_{I_1}, \mathbb{X}_{0,s}\rangle\mathbb{E}[\langle e_{I_2}, \mathbb{X}_{s,t}\rangle].$$

Stone Weierstrass or the universal approximation theorem

Fix a continuous semimartingale X with $X_0 = 0$.

Let $(\widehat{\mathbb{X}}_t^2)_{t\in[0,T]}$ denote the signature of (t, X_t) *truncated* at level 2:

$$\widehat{\mathbb{X}}_t^2 = (1, t, X_t^1, \dots, X_t^d, \int_0^t s \circ ds, \dots, \int_0^t X_s^d \circ dX_s^d)$$

Then every quantity of the form

$$f\Big((\widehat{\mathbb{X}}_t^2)_{t\in[0,T]}\Big)$$

for some continuous map f can be almost surely approximated arbitrarily well on compact sets by objects of the form $\sum_{I\in\mathcal{I}} \lambda_I \langle e_I, \widehat{\mathbb{X}}_T \rangle$, where $\lambda_I \in \mathbb{R}$ and $\mathcal{I}$ contains a finite number of indices I.

A bit of research (joint work with C. Cuchiero and G. Gazzani)

https://doi.org/10.1137/22M1512338

The model

Goal: provide a *good* model for a set of *traded assets* S.

$\rightarrow$*good* $=$ universal, tractable, and easy to calibrate.

Main ingredient: the *market's primary (underlying) process* $\widehat{X}_t := (t, X_t)$.

Requested properties:

- Trajectories of $\widehat{X}$ can be easily generated.
- It is reasonable to assume that:
  - X is d-dimensional continuous semimartingale.
  - $\widehat{X}$ encodes all the randomness of S in a good way, meaning that the paths of S are continuous maps of the paths of $\widehat{X}$.

The model:

$$S_n(\ell)_t := \ell_\emptyset + \sum_{0<|I|\leq n} \ell_I \langle e_I, \widehat{\mathbb{X}}_t \rangle,$$

- $\widehat{\mathbb{X}}$ is the signature of $\widehat{X}$,
- $n \in \mathbb{N}$ is the degree of truncation,
- $\ell_\emptyset, \ell_I \in \mathbb{R}$ are the deterministic coefficients to be found.

See also Perez Arribas, Salvi, Szpruch ('20).

The model: $S_n(\ell)_t := \ell_\emptyset + \sum_{0<|I|\leq n} \ell_I \langle e_I, \widehat{\mathbb{X}}_t \rangle$

Flexibility: From the UAT S_T can be approximated by $S_n(\ell)_t$.

Classical requirements: No arbitrage can easily be guaranteed.

Tractability: Time extended signature of $S_n(\ell)$ can be written as map of $(\ell, \widehat{\mathbb{X}})$.
$\longrightarrow$ Knowing $\mathbb{E}_\mathbb{Q}[\widehat{\mathbb{X}}_T]$, computing an approximation of the price of (path-dependent) options reduces to evaluating a polynomial. Mathematically:

$$\mathbb{E}_\mathbb{Q}[F((S_n(\ell)_t)_{t\in[0,T]})] \approx \mathbb{E}_\mathbb{Q}[L(\widehat{\mathbb{S}}_n(\ell)_T)] = P(\ell, \mathbb{E}_\mathbb{Q}[\widehat{\mathbb{X}}_T]),$$

for some linear map L and some P such that $P(\,\cdot\,, \mathbb{E}_\mathbb{Q}[\widehat{\mathbb{X}}_T])$ is polynomial.

$\longrightarrow$ Formulas for the computations of $\mathbb{E}_\mathbb{Q}[\widehat{\mathbb{X}}_T]$ are available if X is a sufficiently regular Markov (or non Markov) diffusion.

Calibration to option prices

Remark: a call option on S with strike K and maturity T is a contract that pays the random quantity

$$(S_T - K)^+$$

at time T. Its price is given by

$$\mathbb{E}_{\mathbb{Q}}[(S_T - K)^+],$$

where $\mathbb{Q}$ is an equivalent martingale measure, i.e. a probability measure equivalent to the original one, under which S is a local martingale.

Idea: Find parameters ℓ such that the theoretical prices

$$\mathbb{E}_{\mathbb{Q}}[(S_n(\ell)_T - K)^+]$$

matches the prices available on the market, for several T and K.

Good news! $S_n(\ell)$ is a linear model: an update of the parameters does not require new simulations.

Calibration to option prices: the Heston model

- Consider a Heston model (d=2):

$$dS_t = \mu S_t dt + S_t \sqrt{V_t} dW_t^{\mathbb{P},1},$$
$$dV_t = \kappa(\theta - V_t)dt + \xi \sqrt{V_t} dW_t^{\mathbb{P},2}.$$

- Goal: approximate S with $S_3(\ell^*)$, using two $\mathbb{Q}$-Brownian motions as primary underlying process ($\ell^* \in \mathbb{R}^{13}$).
- Test: Compute the implied volatility surface (using montecarlo) under $S_3(\ell^*)$ (orange) and compare it with the Heston's one (blue).

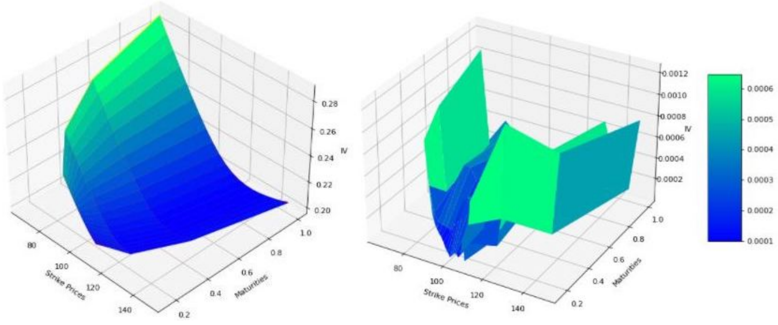

Figure: IVSs and corresponding absolute error with 4 maturities within one year

Calibration to option prices: S&P 500 17.03.2021

- Let S be the stochastic process describing the price of S&P 500 starting at day 17.03.2021.
- Goal: approximate S with $S_3(\ell^*)$, using two $\mathbb{Q}$-Brownian motions as primary underlying process ($\ell^* \in \mathbb{R}^{13}$).
- Test: Compute the implied volatility surface (using montecarlo) under $S_3(\ell^*)$ (orange) and compare it with the market's one (green).

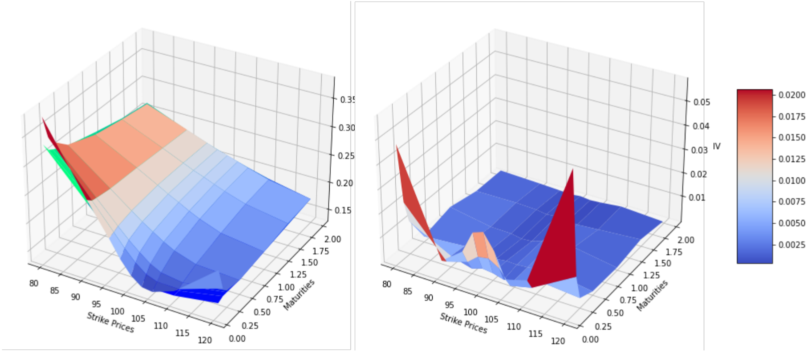

Figure: IVSs and corresponding absolute error with 7 maturities within two years

Conclusions

Conclusions

- We saw that from a mathematical point of view signatures have some extremely interesting properties and deserve to be used in a modeling context.

⇒ $F((X_t)_{t\in[0,T]}) \approx L(\widehat{\mathbb{X}}_T)$ for some linear map L.

- We introduced a linear model based on the signature of an underlying process.

⇒ Flexible: classical models can be approximated arbitrarily well.

⇒ Tractable: since as soon as $\mathbb{E}_{\mathbb{Q}}[\widehat{\mathbb{X}}_T]$ is available, approximated (even path dependent!) option prices are available as polynomials in ℓ. This provides very useful tools for the calibration's procedure.

- We illustrated a calibration method showing the corresponding performances on simulated and real data.

Time expansion for stochastic processes

(joint work with F. Bandi and R. Renò)

https://arxiv.org/pdf/2504.06351

An illustrative example

Let W^1 be the first component of a Brownian motion.

- Consider a stochastic process S admitting the representation

$$S_t = S_0 + \int_0^t c_0(s)ds + \int_0^t c_1(s)dW_s^1.$$

- Suppose that the processes c_0 and c_1 admit the same representation:

$$\begin{aligned} c_0(t) &= c_0(0) + \int_0^t c_{00}(s)ds + \int_0^t c_{10}(s)dW_s^1, \\ c_1(t) &= c_1(0) + \int_0^t c_{01}(s)ds + \int_0^t c_{11}(s)dW_s^1. \end{aligned}$$

- Then

$$\begin{aligned} S_t &= S_0 + \int_0^t \underbrace{c_0(s)}_{=c_0(0)+\int_0^s c_{00}(r)dr+\int_0^s c_{10}(r)dW_r^1} ds + \int_0^t \underbrace{c_1(s)}_{=c_1(0)+\int_0^s c_{01}(r)dr+\int_0^s c_{11}(r)dW_r^1} dW_s^1 \\ &= S_0 + c_0(0)t + c_1(0)W_t^1 + \underbrace{\text{(linear combination of double integrals)}}_{=:\varepsilon_1(t)} \end{aligned}$$

An illustrative example: a further step

$$
\begin{aligned}
S_t &= S_0 + \int_0^t \underbrace{c_0(s)}_{=c_0(0)+\int_0^s c_{00}(r)dr+\int_0^s c_{10}(r)dW_r^1} ds + \int_0^t \underbrace{c_1(s)}_{=c_1(0)+\int_0^s c_{01}(r)dr+\int_0^s c_{11}(r)dW_r^1} dW_s^1, \\
&= S_0 + c_0(0)t + c_1(0)W_t^1 \\
&\quad + \int_0^t \int_0^s c_{00}(r)drds + \int_0^t \int_0^s c_{10}(r)dW_r^1 ds \\
&\quad + \int_0^t \int_0^s c_{01}(r)drdW_s^1 + \int_0^t \int_0^s c_{11}(r)dW_r^1 dW_s^1.
\end{aligned}
$$

- Suppose that the processes c_{00}, c_{01}, c_{10}, and c_{11} admit the same representation:

$$
c_{ij}(s) = c_{ij}(0) + \int_0^s c_{0ij}(r)dr + \int_0^s c_{1ij}(r)dW_r^1.
$$

- Then

$$
\begin{aligned}
S_t &= S_0 + c_0(0)\int_0^t 1ds + c_1(0)\int_0^t 1dW_s^1 \\
&\quad + c_{00}(0)\int_0^t \int_0^s 1drds + c_{10}(0)\int_0^t \int_0^s 1dW_r^1 ds \\
&\quad + c_{01}(0)\int_0^t \int_0^s 1drdW_s^1 + c_{11}(0)\int_0^t \int_0^s 1dW_r^1 dW_s^1 \\
&\quad + (\text{linear combination of triple integrals}) \qquad =: \varepsilon_2(t),
\end{aligned}
$$

An illustrative example: as many steps as we want

Assuming that the procedure can be repeated till depth n and setting $\widehat{W}_t^0 = t$ and $\widehat{W}_t^1 = W_t$ we get

$$S_t = S_0 + \sum_{k=1}^{n} \sum_{(i_1,\dots,i_n)\in\{0,1\}^n} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} 1 d\widehat{W}_{t_1}^{i_1} \cdots d\widehat{W}_{t_n}^{i_n}$$
$$+(\text{linear combination of } n+1 \text{ iterated integrals}) \qquad =: \varepsilon_n(t)$$

Why is this nice?

- $\varepsilon_n(t)$ is an error term in cases of interests. Example: if $t \mapsto \mathbb{E}[c_{i_1,\dots,i_{n+1}}(t)^{2N}]$ is bounded on $[0,\delta]$, for all $m \leq 2N$ we get

$$\mathbb{E}[|\varepsilon_n(t)|^m] \leq Ct^{m(n+1)/2}.$$

- $S_t - \varepsilon_n(t)$ is linear map of a Markovian process.
- The red building blocks are signature's components: multiplying two of them we obtain a linear combination of them... and many other cool properties!

Generator of $\widehat{\mathbb{W}}_t$

The generator $\mathcal{G}$ of $\widehat{\mathbb{W}}_t$ satisfies

$$\mathcal{G} f_c(\widehat{\mathbb{W}}_t) = f'(\langle c, \widehat{\mathbb{W}}_t \rangle)\langle \mathcal{B}(c), \widehat{\mathbb{W}}_t \rangle + \frac{1}{2} f''(\langle c, \widehat{\mathbb{W}}_t \rangle) \underbrace{\langle \Sigma(c), \widehat{\mathbb{W}}_t \rangle^2}_{=\langle \Sigma(c) \star \Sigma(c), \widehat{\mathbb{W}}_t \rangle},$$

where $f_c := f(\langle c, \cdot \rangle)$. In particular,

$$\mathcal{G} f_c(\widehat{\mathbb{W}}_0) = f'(c_\emptyset)\mathcal{B}(c)_\emptyset + \frac{1}{2} f''(c_\emptyset)(\Sigma(c) \star \Sigma(c))_\emptyset$$

and thus given $S_t = \langle c, \widehat{\mathbb{W}}_t \rangle + \varepsilon_n(t)$ we get

$$\mathbb{E}[f(S_t)] = f(S_0) + \left(f'(S_0)\mathcal{B}(c)_\emptyset + \frac{1}{2} f''(S_0)(\Sigma(c) \star \Sigma(c))_\emptyset \right) t + o(t).$$

Summarizing

One can show that $\mathcal{G}$ is mapping functions of the form $f(\langle c,\cdot\rangle)(\langle d,\cdot\rangle)$ to linear combination of functions of the form

$$f^{(k)}(\langle c,\cdot\rangle)(\langle \mathcal{G}^k_c(d),\cdot\rangle)$$

for some bilinear operator $\mathcal{G}^k$ with $k = 0, 1, 2$.

Theorem

Consider an n-times $\widehat{W}$-differentiable process $(S_t)_{t\in[0,T]}$ with expansion

$$S_t = \langle c, \widehat{\mathbb{W}}_t\rangle + \varepsilon_n(t).$$

Then for each $f \in C^{n+1}(\mathbb{R})$ it holds

$$\mathbb{E}[f(S_t)] = f(S_0) + \sum_{\ell=1}^{\lceil n/2\rceil} \frac{1}{\ell!}\left(\sum_{k_1,\dots,k_\ell=0}^{2} f^{(k_1+\dots+k_\ell)}(S_0)\mathcal{G}_{c,k_1,\dots,k_\ell}(\emptyset)_\emptyset \right) t^\ell + o(t^{n/2}),$$

where $\mathcal{G}_{c,k_1,\dots,k_n}(d) = \mathcal{G}^{k_n}_c(\cdots(\mathcal{G}^{k_1}_c(d)))$.

Good news: Bilinear maps are easy to code!
Given $f(S_0)$, $f'(S_0)\dots$, $f^{(n+1)}(S_0)$ the coefficients can be computed by a computer.

Example

Suppose that S is 2 times $\widehat{W}$-differentiable with

$$\begin{aligned} S_t &= S_0 + \int_0^t \underbrace{c_0(s)}_{=c_0(0)+\int_0^s c_{00}(r)dr+\int_0^s c_{10}(r)dW_r^1} ds + \int_0^t \underbrace{c_1(s)}_{=c_1(0)+\int_0^s c_{01}(r)dr+\int_0^s c_{11}(r)dW_r^1} dW_s^1 \\ &= S_0 + c_0(0)t + c_1(0)W_t^1 + \ldots + c_{11}(0)\int_0^t W_s^1 dW_s^1 + \varepsilon_2(t). \end{aligned}$$

Then applying the theorem for $n = 2$ yields

$$\mathbb{E}[e^{iu(S_t - S_0)^2}] = 1 + iu(c_1(0))^2 t + o(t).$$

Joint calibration of SPX and VIX options

(joint work with C. Cuchiero, G. Gazzani and J. Möller)

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The model

We consider a model where the dynamics of the S&P-500 index and the corresponding volatility process are given by

$$\begin{aligned} dS_t &= S_t \sigma_t^S dB_t \\ \sigma_t^S &= \ell_\emptyset + \sum_{0<|I|\leq n} \ell_I \langle e_I, \widehat{\mathbb{X}}_t \rangle, \end{aligned}$$

where

- $B = (B_t)_{t\geq 0}$ is a one-dimensional Brownian motion
- $n \in \mathbb{N}$
- $X = (X^1, \dots, X^d)$ is a d-dimensional continuous semimartingale. Denoting $Z = (X, B)$, then the correlation matrix between X and B is given by

$$\rho_{i,j} = \frac{[Z^i, Z^j]}{\sqrt{[Z^i]}\sqrt{[Z^j]}} \in [-1, 1],$$

  for all $i, j = 1, \dots, d+1$, where $[\cdot, \cdot]$ denotes the quadratic covariation.
- $\ell := \{\ell_I \in \mathbb{R} : |I| \leq n\}$ the collection of parameters of the model.

A reasonable choice for the primary process

Our choice for the primary process goes back to the good old polynomial diffusions: we assume that

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dW_t,$$

where

- b, a are polynomial of order one and two, respectively.
- $W = (W_t)_{t\geq 0}$ is a d-dimensional Brownian motion.

Why? Because they are a wide class and the truncated time extended signature

$$\widehat{\mathbb{X}}_t^{\leq N} := (1, \langle e_0, \widehat{\mathbb{X}}_t \rangle, \ldots, \langle e_{(d,\ldots,d)}, \widehat{\mathbb{X}}_t \rangle)$$

of $(X_t)_{t\geq 0}$ is a polynomial process too. We can thus compute its conditional moments by means of a matrix exponential.

But why do we care? Because the VIX index on S can be re-written as

$$VIX_T = \sqrt{\frac{1}{\Delta}\mathbb{E}\left[\int_T^{T+\Delta} (\sigma_t^S)^2 dt | \mathcal{F}_T\right]},$$

and $\int_T^{T+\Delta} (\sigma_t^S)^2 dt$ is a polynomial (of degree 1) in $\widehat{\mathbb{X}}_T^{2n+1}$ and $\widehat{\mathbb{X}}_{T+\Delta}^{2n+1}$.

Even more:

- Jumps can be included.
- Non-truncated signature can be studied.
- Other powerful techniques can be employed!
  What about results from complex analysis for power series?
  What about duality theory?
- More challenging joint calibrations can be considered: what about the VXX?
- Other algebras can be considered: are polynomials the only way?

... and more!

Cass, Fermanian,Kalsi, Kidger, Horvath, Ferrucci, Lyons, Ni, Oberhauser, Perez Arribas, Sabate-Vidales, Salvi, Szpruch, Yang...

Abi Jaber, Gassiat, Gérard, Huang, Pannier,...

Cuchiero, Gazzani, Gonon, Guo, Möller, Primavera, Teichmann,...

Bayer, Bank, Dos Reis, Friz, Hager, Harang, Riedel, Schoenmakers, Tapia,...

Bandi, Grasselli, Renò, Stanghellini...

Ben Arous, Chen, Kloeden, Platen,...

Thank you for your attention!